

ON THE CLUSTER NATURE AND QUANTIZATION OF GEOMETRIC R -MATRICES

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ABSTRACT. We define cluster R -matrices as sequences of mutations in triangular grid quivers on a cylinder, and show that the affine geometric R -matrix of symmetric power representations can be obtained from our cluster R -matrix. A quantization of the affine geometric R -matrix is defined, compatible with the cluster structure. We construct invariants of the quantum affine geometric R -matrix as quantum loop symmetric functions.

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1. INTRODUCTION

The R -matrix of a quantum group, or that of a representation of a quantum group, plays a vital role in the connection between representation theory and the theory of integrable systems. The transformations appearing in this paper arise from R -matrices of Kirillov-Reshetikhin modules of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}'_n)$, from which *affine combinatorial R -matrices* are obtained as crystal limits, see [KKMMNN]. The combinatorial R -matrix is a bijection $R_{\text{comb}}: B \otimes B' \rightarrow B' \otimes B$ between tensor products of Kirillov-Reshetikhin crystals B and B' . In this paper, we focus on the case where $B = B_\ell$ (resp. $B' = B_{\ell'}$) is the crystal graph of the symmetric tensor module $\text{Sym}^\ell \mathbb{C}^n$ (resp. $\text{Sym}^{\ell'} \mathbb{C}^n$).

In this case, the bijection R_{comb} can be interpreted as the restriction of a piecewise-linear map $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ to a finite subset of lattice points. This piecewise-linear map is the tropicalization of a rational morphism $R: (\mathbb{C}^*)^n \times (\mathbb{C}^*)^n \rightarrow (\mathbb{C}^*)^n \times (\mathbb{C}^*)^n$ called the *geometric R -matrix* [KNY, KNO, Eti, LP12], which plays an important role in

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the theory of affine geometric crystals. We let $\mathbf{p} = (p_1, \dots, p_n)$ and $\mathbf{q} = (q_1, \dots, q_n)$ be coordinates on $(\mathbb{C}^*)^n \times (\mathbb{C}^*)^n$. Then the geometric R -matrix is an isomorphism $R: \mathbb{C}(\mathbf{p}, \mathbf{q}) \rightarrow \mathbb{C}(\mathbf{p}, \mathbf{q})$ of fields. It satisfies the Yang-Baxter relation, giving an action of the symmetric group \mathfrak{S}_m on $\mathbb{C}(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_m)$ by rational transformations, where $\mathbf{q}_j = (q_{j,1}, \dots, q_{j,n}) \in (\mathbb{C}^*)^n$.

1.1. Quantum geometric R -matrix. Our first main result is the construction of a quantization of the rational map R , called the *quantum geometric R -matrix* R^ϵ . The quantum geometric R -matrix is an isomorphism $R^\epsilon: \mathbb{C}_\epsilon\langle \mathbf{p}, \mathbf{q} \rangle \rightarrow \mathbb{C}_\epsilon\langle \mathbf{p}, \mathbf{q} \rangle$ between the skew fraction fields of certain quantum tori. Theorem 5.3 states that, like the classical, combinatorial, and geometric R -matrices, the quantum geometric R -matrix also satisfies a Yang-Baxter relation. We may summarize the four types of R -matrix transformations as follows.

	Transformation	Acts on
(quantum) R -matrix	$\mathbb{C}(q)$ -linear	$U_q(\hat{\mathfrak{sl}}'_n)$ -representations $V \otimes V'$
combinatorial R -matrix	piecewise linear	$\hat{\mathfrak{sl}}_n$ -crystals $B \otimes B'$
geometric R -matrix	rational map	affine geometric crystals $X \times X'$
quantum geometric R -matrix	skew field map	quantum geometric crystals?

The quantum geometric R -matrix generates an action of the symmetric group \mathfrak{S}_m on a skew field $\mathcal{Q}_\epsilon := \mathbb{C}_\epsilon\langle \mathbf{q}_1, \dots, \mathbf{q}_m \rangle$. We give a ‘symmetric function’ description of certain elements in the invariant subfield $\mathcal{Q}_\epsilon^{\mathfrak{S}_m}$. Namely, we define quantum loop elementary symmetric functions (Corollary 6.3), quantum loop Schur functions (Theorem 6.6) and quantum cylindric loop Schur functions (Theorem 6.11), and prove that they are invariants. This generalizes results from [Yam, LP12, LPS].

Quantum loop Schur functions and quantum cylindric loop Schur functions have two closely related descriptions: (1) as the (noncommutative) generating function of certain semistandard Young tableaux or semistandard cylindric tableaux, and (2) as the (noncommutative) generating function of families of *highway paths* in a network $N_{n,m}$ on the cylinder, previously studied in [LP13]. We quantize an argument from [LP13] to show that the quantum geometric R -matrix can be obtained as a composition of certain local transformations of the cylindric network $N_{n,m}$.

1.2. Cluster R -matrix. We discovered the quantum geometric R -matrix after realizing the geometric R -matrix as a composition of cluster mutations for a certain ‘triangular grid’ quiver $Q_{n,m}$ on a cylinder. Our second main result is the construction of this *cluster R -matrix*. The cylindric quiver $Q_{n,m}$ and the cylindric network $N_{n,m}$ are related in a similar way to how the quiver of the cluster algebra for a Bruhat cell is related to the wiring diagram of a reduced word [BFZ05].

The cluster R -matrix is defined to be the composition of $2n - 2$ cluster mutations of $Q_{n,m}$ which (rather non-trivially) sends $Q_{n,m}$ back to itself, and furthermore satisfies a Yang-Baxter relation (Theorem 3.2). Further we consider a decorated quiver $\tilde{Q}'_{n,m}$ of $Q_{n,m}$ with frozen variables, and the natural coordinates \mathbf{q}_i of the geometric R -matrix

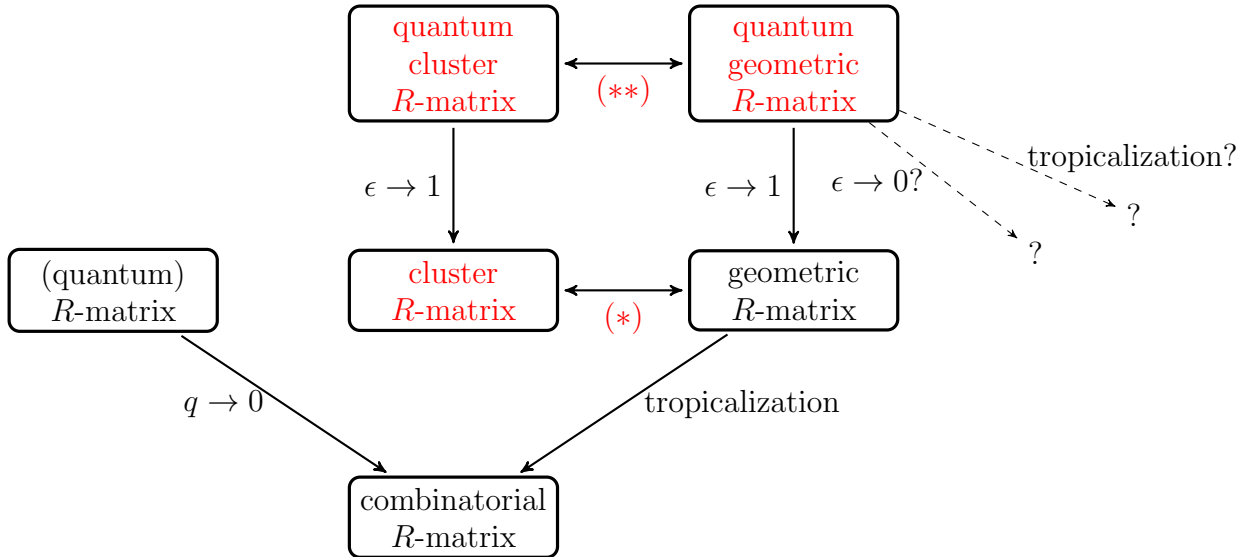
are related to the cluster variables x and frozen variables X of the corresponding cluster algebra by a relation of the form

$$q = \frac{xx'}{x''x'''}X. \quad (1.1)$$

This is reminiscent of τ -function substitutions in the theory of integrable systems, and of formulae occurring in the chamber ansatz [BFZ96].

Broadly speaking, there are two notions of quantization for cluster algebras. Berenstein and Zelevinsky [BZ] have defined quantum cluster algebras that quantize cluster mutation. This notion however depends on additional choices. Fock and Goncharov [FG09a] have defined a canonical quantization of mutation of cluster y -seeds, or equivalently, of the coefficient dynamics. Working with the same sequence of mutations of $Q_{n,m}$, but this time with quantum cluster y -seeds, we construct the *quantum cluster R -matrix*. We show that the quantum cluster R -matrix satisfies a Yang-Baxter relation (Theorem 7.6) and that it is compatible with the quantum geometric R -matrix (Theorem 7.9) via an embedding $\phi: \mathcal{Y}'_\epsilon \rightarrow \mathcal{Q}_\epsilon$, where \mathcal{Y}'_ϵ is a subfield of the quotient skew field of the quantum torus of a y -seed.

The following diagram is an overview of the relations between the different R -matrices. We denote our results in this paper in red. The arrow $(*)$ is established by the ‘ τ -function substitution’ (1.1). The arrow $(**)$ is established by the embedding of skew fields ϕ .



Outline. In §2 we recall the necessary background on cluster algebras and their properties. In §3 we introduce the cluster R -matrix as a sequence of mutations on a triangular grid quiver on a cylinder. First we study the case of a simple quiver without frozen vertices, then we consider the case of the decorated quiver with frozen vertices. In both cases we prove that the R -matrices generate the symmetric group action on the field of cluster variables. In §4 we show how the geometric R -matrix is induced by the cluster R -matrix. In §5, we define a skew field \mathcal{Q}_ϵ and the quantum geometric R -matrices acting on it. We state the result that the quantum geometric R -matrices generate a symmetric group action on \mathcal{Q}_ϵ . In §6 we define some distinguished invariants of the quantum geometric R -matrix. We use a network model on a cylinder to prove the invariance. In §7 we define the quantum cluster R -matrices, using Fock-Goncharov

quantization of cluster y -seeds. The quantum cluster R -matrices generate the symmetric group action on the skew field of quantum y -variables. We then show that the quantum cluster R -matrix is compatible with the quantum geometric R -matrix. In particular, we use the Yang-Baxter relation for the former to deduce the Yang-Baxter relation for the latter. In §8 we collect the proofs of some technical results in §3.

2. CLUSTER ALGEBRA PRELIMINARIES

2.1. Seeds and mutation. We recall the notion of seeds and mutation of cluster algebras from [FZ02]. Our cluster algebras will be skew-symmetric.

Fix a finite set I , and let N be its cardinality. Let (B, \mathbf{x}) denote a seed, where $\mathbf{x} = (x_i)_{i \in I}$ is a collection of cluster variables and $B = (b_{ij})_{i,j \in I}$ is an $N \times N$ skew-symmetric integral matrix called the exchange matrix. The variables x_i are assumed to be algebraically independent elements of some ambient field. For $k \in I$, the mutation μ_k of a seed (B, \mathbf{x}) is defined to be $(\tilde{B}, \tilde{\mathbf{x}}) = \mu_k(B, \mathbf{x})$, where

$$\tilde{x}_i = \begin{cases} x_i & i \neq k, \\ \frac{\prod_{j: b_{jk} > 0} x_j^{b_{jk}} + \prod_{j: b_{jk} < 0} x_j^{-b_{jk}}}{x_k} & i = k, \end{cases} \quad (2.1)$$

$$\tilde{b}_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases} \quad (2.2)$$

Mutation is an involutive operation, and μ_i and μ_j commute when $b_{ij} = 0$. For an exchange matrix B , we have a quiver $Q = Q(B)$ with vertices identified with the set I , without 1-cycles and 2-loops, satisfying

$$b_{ij} = \# \{\text{arrows from } i \text{ to } j\} - \# \{\text{arrows from } j \text{ to } i\},$$

for $i, j \in I$. The mutation $\mu_k(Q) = Q(\mu_k(B))$ of a quiver $Q = Q(B)$ is obtained by the following procedure:

- (1) for every pair of directed edges $i \rightarrow k$ and $k \rightarrow j$, we add a directed edge $i \rightarrow j$;
- (2) reverse all directed edges going into or coming out of k ;
- (3) remove any 2-cycles in pairs $\{i \rightarrow j, j \rightarrow i\}$.

Often one is in the situation where some vertices of a quiver Q (resp. some cluster variables) are never mutated. We call such vertices (resp. variables) frozen, and the other vertices (resp. variables) are called mutable. In practice, edges between frozen vertices can be disregarded (resp. columns of the exchange matrix B indexed by frozen vertices can be ignored).

2.2. Seeds with coefficients and y -seeds. We recall the notion of seeds with coefficients and the y -seeds from [FZ07].

Let $(\mathbb{P}, \cdot, \oplus)$ be a semifield, that is, an abelian multiplicative group with an addition \oplus which is commutative, associative, and distributive with respect to the multiplication \cdot in \mathbb{P} . Let $\mathbb{Q}\mathbb{P}$ denote the fraction field of the group ring $\mathbb{Z}\mathbb{P}$. Let \mathcal{F} denote a rational function field over $\mathbb{Q}\mathbb{P}$ generated by some algebraically independent elements.

A seed with coefficients $(B, \mathbf{x}, \mathbf{y})$ consists of a $N \times N$ skew-symmetric integral matrix B , a collection $\mathbf{x} = (x_i)_{i \in I}$ of algebraically independent elements in \mathcal{F} , and a collection $\mathbf{y} = (y_i)_{i \in I}$ of elements in \mathbb{P} . Each y_i is called a coefficient, or a y -variable. For $k \in I$, the mutation $\mu_k(B, \mathbf{x}, \mathbf{y}) = (\tilde{B}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is defined as follows. The matrix \tilde{B} is defined as before, and we have

$$\tilde{x}_i = \begin{cases} x_i & i \neq k, \\ \frac{y_k \prod_{j: b_{jk} > 0} x_j^{b_{jk}} + \prod_{j: b_{jk} < 0} x_j^{-b_{jk}}}{(1 \oplus y_k)x_k} & i = k, \end{cases} \quad (2.3)$$

$$\tilde{y}_i = \begin{cases} y_k^{-1} & i = k, \\ y_i (1 \oplus y_k^{-1})^{-b_{ki}} & i \neq k, \ b_{ki} \geq 0, \\ y_i (1 \oplus y_k)^{-b_{ki}} & i \neq k, \ b_{ki} \leq 0. \end{cases} \quad (2.4)$$

A y -seed (B, \mathbf{y}) and its mutations are defined by ignoring the cluster variables \mathbf{x} of the seed $(B, \mathbf{x}, \mathbf{y})$.

Two particular semifields will be important to us: the universal semifield $\mathbb{P}_{\text{univ}}(\mathbf{y}^0)$ and the tropical semifield $\mathbb{P}_{\text{trop}}(\mathbf{y}^0)$, generated by a set $\mathbf{y}^0 = (y_i^0)_{i \in I}$ of initial y -variables. The universal semifield $\mathbb{P}_{\text{univ}}(\mathbf{y}^0)$ is the subset of all rational functions f in y_i^0 ($i \in I$) such that $f \in \mathbb{P}_{\text{univ}}(\mathbf{y}^0)$ has a subtraction-free rational expression, endowed with the ordinary multiplication and addition ‘+’ of rational functions. The tropical semifield $\mathbb{P}_{\text{trop}}(\mathbf{y}^0)$ is an abelian multiplicative group on y_i^0 ($i \in I$) endowed with the addition \oplus defined by

$$\prod_{i \in I} (y_i^0)^{n_i} \oplus \prod_{i \in I} (y_i^0)^{m_i} = \prod_{i \in I} (y_i^0)^{\min(n_i, m_i)},$$

for $n_i, m_i \in \mathbb{Z}$.

We define a surjection $\pi: \mathbb{P}_{\text{univ}}(\mathbf{y}^0) \rightarrow \mathbb{P}_{\text{trop}}(\mathbf{y}^0)$ given by

$$f \mapsto [f] := \prod_{i \in I} (y_i^0)^{n_i}.$$

Here $[f]$ is called the ‘principal coefficient’ of f , determined uniquely by writing

$$f = \prod_{i \in I} (y_i^0)^{n_i} \cdot \frac{f_1}{f_2}$$

where f_1 and f_2 are polynomials of the y_i^0 with non-zero constant terms. It is easy to check that π is a homomorphism of semifields.

2.3. Fock-Goncharov quantization. We introduce the quantization of the y -seeds, following [FG09a].

Let (B, \mathbf{y}) be a quantum y -seed, where $B = (b_{ij})_{i, j \in I}$ be an N by N skew-symmetric integral matrix as in the classical case, and $\mathbf{y} = (y_i)_{i \in I}$ generate the quantum torus $\mathbb{C}_\epsilon[\mathbf{y}]$, the noncommutative $\mathbb{C}[\epsilon]$ -algebra with relations given by

$$y_i y_j = \epsilon^{2b_{ji}} y_j y_i.$$

Let $\mathcal{Y}_\epsilon = \mathbb{C}_\epsilon\langle \mathbf{y} \rangle$ be the skew fraction field of $\mathbb{C}_\epsilon[\mathbf{y}]$. Let μ_k^ϵ be the quantum mutation of a y -seed (B, \mathbf{y}) , where $(\tilde{B}, \tilde{\mathbf{y}}) = \mu_k^\epsilon(B, \mathbf{y})$ is given by

$$\tilde{y}_i = \begin{cases} y_k^{-1} & \text{if } i = k, \\ y_i \prod_{m=1}^{b_{ki}} (1 + \epsilon^{2m-1} y_k^{-1})^{-1} & \text{if } i \neq k, \ b_{ki} \geq 0, \\ y_i \prod_{m=1}^{-b_{ki}} (1 + \epsilon^{2m-1} y_k) & \text{if } i \neq k, \ b_{ki} \leq 0, \end{cases} \quad (2.5)$$

and \tilde{B} is the same as in the classical case (2.2). Let $\tilde{\mathcal{Y}}_\epsilon$ be the skew fraction field generated by the \tilde{y}_i with the ϵ -commutativity determined by \tilde{B} . It is known that μ_k^ϵ induces a morphism of the skew fraction fields $\tilde{\mathcal{Y}}_\epsilon \rightarrow \mathcal{Y}_\epsilon$. Similarly to the classical case, μ_i^ϵ is involutive, and μ_i^ϵ and μ_j^ϵ commute when $b_{ij} = 0$. By setting $\epsilon = 1$, (2.5) reduces to (2.4).

2.4. Periods of cluster mutations. The permutation group $\mathfrak{S}(I)$ naturally acts on y -seeds by permuting the variables and the rows and columns of the exchange matrix. For $\sigma \in \mathfrak{S}(I)$, a σ -period of a y -seed is a sequence of mutations that sends a seed back to itself up to σ ; see [Nak] for more details. The following theorem gives an important and useful relation among the classical, tropical, and quantum cluster y -seeds.

Theorem 2.1. *For a positive integer k and an I -sequence $\mathbf{i} := (i_1, i_2, \dots, i_k) \in I^k$, define a sequence of mutations $\mu_{\mathbf{i}} := \mu_{i_k} \circ \mu_{i_{k-1}} \circ \dots \circ \mu_{i_1}$. Let $\sigma \in \mathfrak{S}(I)$ be a permutation. Then the following three statements are equivalent:*

- (1) *Assume that the semifield is $\mathbb{P}_{\text{univ}}(\mathbf{y})$. The I -sequence \mathbf{i} is a σ -period of the y -seed (B, \mathbf{y}) , i.e., $\mu_{\mathbf{i}}(B, \mathbf{y}) = \sigma(B, \mathbf{y})$.*
- (2) *Assume that the semifield is $\mathbb{P}_{\text{trop}}(\mathbf{y})$. The I -sequence \mathbf{i} is a σ -period of the tropical y -seed (B, \mathbf{y}) , i.e., $\mu_{\mathbf{i}}(B, \mathbf{y}) = \sigma(B, \mathbf{y})$.*
- (3) *The I -sequence \mathbf{i} is a σ -period of the quantum y -seed (B, \mathbf{y}) , i.e., $\mu_{\mathbf{i}}^\epsilon(B, \mathbf{y}) = \sigma(B, \mathbf{y})$.*

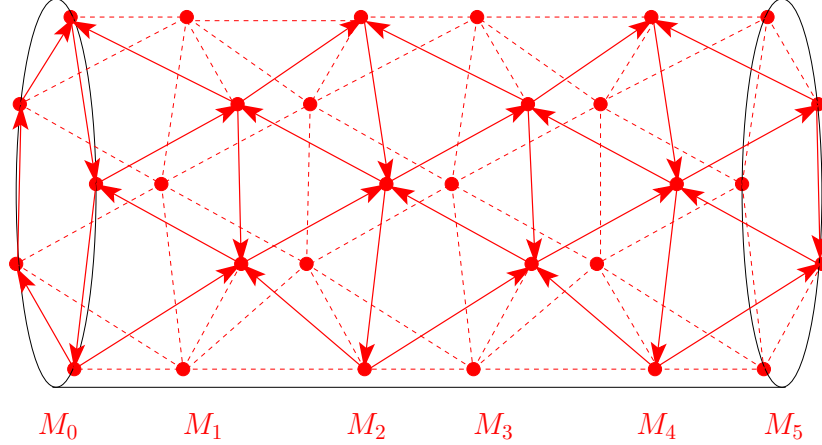
Proof. The claim (1) \Leftrightarrow (3) is proved in [FG09b, Lemma 2.22] when the matrix B is non-degenerate, and generalized to degenerate B in [KN, Proposition 3.4]. The claim (2) \Rightarrow (1) is included in [IIKKN, Theorem 5.1]. The claim (1) \Rightarrow (2) is deduced from the map π . \square

We will also use the following result from [IIKKN, Theorem 5.1].

Theorem 2.2. *Let $\sigma \in \mathfrak{S}(I)$ be a permutation and \mathbf{i} be an I -sequence as in Theorem 2.1. Suppose that \mathbb{P} is the semifield $\mathbb{P}_{\text{trop}}(\mathbf{y})$, and \mathbf{i} is a σ -period of the seed (B, \mathbf{y}) . Then \mathbf{i} is a σ -period of the seed $(B, \mathbf{x}, \mathbf{y})$, i.e., $\mu_{\mathbf{i}}(B, \mathbf{x}, \mathbf{y}) = \sigma(B, \mathbf{x}, \mathbf{y})$.*

3. CLUSTER R -MATRIX

3.1. Triangular grid quivers on a cylinder. We consider triangular grid quivers on a cylinder, as in Figure 1. Let $Q_{n,m}$ denote such a quiver with n vertices on each of the two boundary circles and $m+1$ parallel vertical cycles M_0, M_1, \dots, M_m that go around the cylinder, each with n vertices. See Figure 1 for the example of $Q_{5,5}$, where the 6 vertical cycles M_0, \dots, M_5 are arranged from left to right.

FIGURE 1. The quiver $Q_{5,5}$.

3.2. Definition of the simple cluster R -matrix. Let $(\mathbf{x}^-, \mathbf{x}, \mathbf{x}^+)$ be a tuple of $3n$ -variables x_i^-, x_i, x_i^+ where $i \in \mathbb{Z}/n\mathbb{Z}$. The *simple cluster R -matrix* is the rational transformation $R_{\mathbf{x}}: \mathbb{C}(\mathbf{x}^-, \mathbf{x}, \mathbf{x}^+) \rightarrow \mathbb{C}(\mathbf{x}^-, \mathbf{x}, \mathbf{x}^+)$ given by

$$R_{\mathbf{x}}(x_i) = \frac{\sum_{j=1}^n x_{j+1}^- \left(\prod_{\ell=j+2}^{j-1} x_{\ell} \right) x_j^+}{\prod_{j \neq i} x_j}, \quad R_{\mathbf{x}}(x_i^-) = x_i^-, \quad R_{\mathbf{x}}(x_i^+) = x_i^+. \quad (3.1)$$

For example, if $n = 4$, we have

$$R_{\mathbf{x}}(x_1) = \frac{x_2^- x_3 x_4 x_1^+ + x_3^- x_4 x_1 x_2^+ + x_4^- x_1 x_2 x_3^+ + x_1^- x_2 x_3 x_4^+}{x_2 x_3 x_4}.$$

Let Q be a quiver. For a vertex $v \in Q$, we have a cluster variable x_v . Let $\mathcal{F}(Q) = \mathbb{C}(x_v \mid v \in Q)$ be the field generated by all the cluster variables indexed by vertices of Q . In this section we take $\mathcal{F} = \mathcal{F}(Q_{n,m})$. Fix $1 \leq i \leq m-1$. Let (M^-, M, M^+) denote the three cycles M_{i-1}, M_i, M_{i+1} . Let the vertices of M (resp. M^-, M^+) be denoted $i \in \mathbb{Z}/n\mathbb{Z}$ (resp. i^-, i^+). The edges to and from i are given by

$$i \rightarrow i+1, \quad i \rightarrow i^-, \quad i \rightarrow i-1^+, \quad i \leftarrow i-1, \quad i \leftarrow i+1^-, \quad i \leftarrow i^+. \quad (3.2)$$

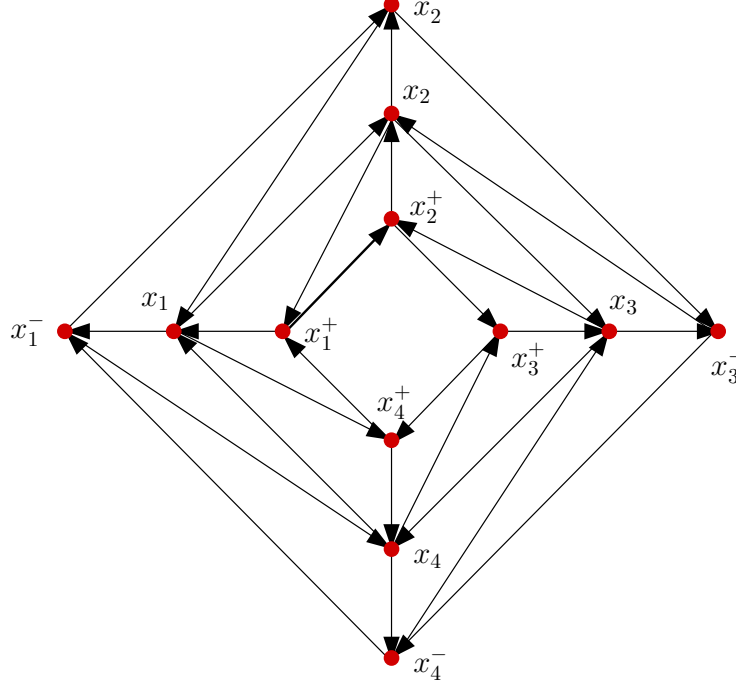
Denote the corresponding cluster variables on M^-, M, M^+ by x_i^-, x_i, x_i^+ respectively. We then have a rational transformation $R_M = R_{\mathbf{x}}: \mathcal{F} \rightarrow \mathcal{F}$ acting on the cluster variables $(\mathbf{x}^-, \mathbf{x}, \mathbf{x}^+)$ and fixing all cluster variables not contained in $M^- \cup M \cup M^+$.

Figure 2 shows all edges of $Q_{n,m}$ incident with a closed vertical cycle M , where $n = 4$. Denote by μ_i the mutation at the vertex i of M , and $s_{i,j}$ the permutation of the vertices i and j on M . For $j \in \mathbb{Z}/n\mathbb{Z}$, we define a sequence of $2n-2$ mutations and a permutation by

$$\mathcal{R}_{M,j} := s_{j-2,j-1} \circ (\mu_j \circ \mu_{j+1} \cdots \circ \mu_{j-4} \circ \mu_{j-3}) \circ (\mu_{j-1} \circ \mu_{j-2} \circ \cdots \circ \mu_{j+1} \circ \mu_j). \quad (3.3)$$

The operation $\mathcal{R}_{M,j}$ acts both on the quiver $Q_{n,m}$, sending it to $\mathcal{R}_{M,j}(Q_{n,m})$, and acts as a rational transformation of \mathcal{F} .

Theorem 3.1. *For any $j \in \mathbb{Z}/n\mathbb{Z}$, we have $\mathcal{R}_{M,j}(Q_{n,m}, \mathbf{x}_{Q_{n,m}}) = (Q_{n,m}, R_M(\mathbf{x}_{Q_{n,m}}))$.*

FIGURE 2. Adjacent vertical closed cycles M^- , M , M^+ .

Theorem 3.2. *The $m-1$ transformations $R_{M_1}, R_{M_2}, \dots, R_{M_{m-1}}: \mathcal{F} \rightarrow \mathcal{F}$ satisfy $R_{M_i} R_{M_{i+1}} R_{M_i} = R_{M_{i+1}} R_{M_i} R_{M_{i+1}}$ and $R_{M_i}^2 = \text{id}$. Furthermore, for i, j such that $|i - j| > 1$ we have $R_{M_i} R_{M_j} = R_{M_j} R_{M_i}$. Thus R_{M_i} generates an action of the symmetric group \mathfrak{S}_m on \mathcal{F} .*

See §8.1 and §8.2 for the proofs.

3.3. Definition of the cluster R -matrix. To the $3n$ variables $(\mathbf{x}^-, \mathbf{x}, \mathbf{x}^+)$, we add $5n$ additional variables denoted $X_{j,j+1}, X_{j^+,j}, X_{j,j^-}, X_{j+1^-,j}, X_{j+1,j^+}$, where $j \in \mathbb{Z}/n\mathbb{Z}$. Denote by \mathbf{X} the set of these new variables, and by $\mathbb{C}(\mathbf{x}, \mathbf{X})$ the rational function field in these variables. The *cluster R -matrix* is the rational transformation $\tilde{R}_{\mathbf{x}}: \mathbb{C}(\mathbf{x}, \mathbf{X}) \rightarrow \mathbb{C}(\mathbf{x}, \mathbf{X})$ given by

$$\tilde{R}_{\mathbf{x}}(x_i) = \frac{\sum_{j=1}^n x_{j+1}^- \left(\prod_{\ell=j+2}^{j-1} x_{\ell} \right) x_j^+ X_{j,j+1} \left(\prod_{\ell=j+1}^{i-1} X_{\ell^+, \ell} X_{\ell+1^-, \ell} \right) \left(\prod_{\ell=i+1}^j X_{\ell, \ell^-} X_{\ell, \ell-1^+} \right)}{\prod_{j \neq i} x_j} \quad (3.4)$$

$$\tilde{R}_{\mathbf{x}}(x_i^-) = x_i^-, \quad \tilde{R}_{\mathbf{x}}(x_i^+) = x_i^+,$$

and

$$\begin{aligned} \tilde{R}_{\mathbf{x}}(X_{i^+,i}) &= X_{i+1,i+1^-}, & \tilde{R}_{\mathbf{x}}(X_{i+1,i+1^-}) &= X_{i^+,i}, \\ \tilde{R}_{\mathbf{x}}(X_{i,i-1^+}) &= X_{i^-,i-1}, & \tilde{R}_{\mathbf{x}}(X_{i^-,i-1}) &= X_{i,i-1^+}, \\ \tilde{R}_{\mathbf{x}}(X_{i,i+1}) &= X_{i,i+1}. \end{aligned} \quad (3.5)$$

For example, if $n = 4$, we have

$$\begin{aligned} x_2 x_3 x_4 \tilde{R}_x(x_1) = & x_2^- x_3 x_4 x_1^+ X_{1,2} X_{2+,2} X_{3-,2} X_{3+,3} X_{4-,3} X_{4+,4} X_{1-,4} \\ & + x_3^- x_4 x_1 x_2^+ X_{2,3} X_{3+,3} X_{4-,3} X_{4+,4} X_{1-,4} X_{2,2-} X_{2,1+} \\ & + x_4^- x_1 x_2 x_3^+ X_{3,4} X_{4+,4} X_{1-,4} X_{2,2-} X_{2,1+} X_{3,3-} X_{3,2+} \\ & + x_1^- x_2 x_3 x_4^+ X_{4,1} X_{2,2-} X_{2,1+} X_{3,3-} X_{3,2+} X_{4,4-} X_{4,3+}. \end{aligned}$$

To each arrow $a \rightarrow a'$ in the quiver $Q_{n,m}$ associate a new (frozen) vertex $v_{aa'}$, connecting it to already existing vertices of the quiver via arrows $a' \rightarrow v_{aa'}$ and $v_{aa'} \rightarrow a$. Denote the resulting enriched quiver by $\tilde{Q}_{n,m}$. Associate to each of the new vertices $v_{a,a'}$ a frozen variable $X_{a,a'}$, and let $\tilde{\mathcal{F}} = \mathcal{F}(\tilde{Q}_{n,m})$ be the rational function field generated by all the cluster variables, including the new frozen variables.

As before, let (M^-, M, M^+) denote the three adjacent cycles M_{i-1}, M_i, M_{i+1} . The cycle M is connected to vertices from the two adjacent cycles M^- and M^+ . Let n be the length of M , and denote the vertices on M^-, M, M^+ as i^-, i, i^+ respectively, where i takes values $\text{mod } n$ and the arrows around i are now

$$i \rightarrow i+1, \quad i \rightarrow i^-, \quad i \rightarrow i-1^+, \quad i \leftarrow i-1, \quad i \leftarrow i+1^-, \quad i \leftarrow i^+,$$

$$i \rightarrow v_{i-1,i}, \quad i \rightarrow v_{i+1-,i}, \quad i \rightarrow v_{i+,i}, \quad i \leftarrow v_{i,i+1}, \quad i \leftarrow v_{i,i-}, \quad i \leftarrow v_{i,i-1+}.$$

We then have a rational transformation $\tilde{R}_M = \tilde{R}_x: \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}}$ acting on the mutable variables on $M^- \cup M \cup M^+$, and permuting some of the frozen variables; the transformation fixes all the variables not appearing in (3.4) or (3.5).

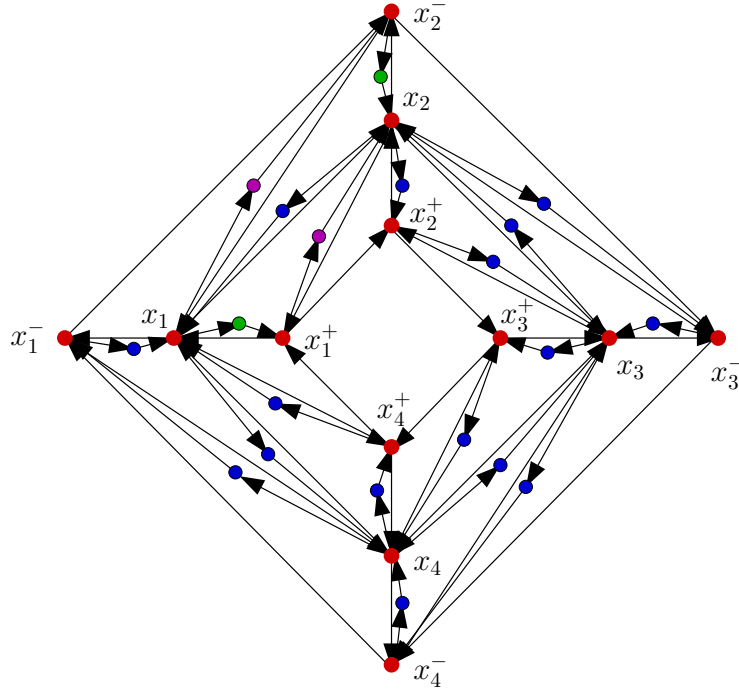


FIGURE 3. Adjacent vertical closed cycles M^-, M, M^+ with frozen vertices added. The cluster R -matrix \tilde{R}_M swaps some of the frozen variables, such as $X_{1+,1}$ with $X_{2,2-}$ (shown in green), and $X_{2,1+}$ with $X_{2-,1}$ (shown in purple).

For $j \in \mathbb{Z}/n\mathbb{Z}$, we define a sequence of $2n - 2$ mutations and $2n + 1$ permutations by

$$\begin{aligned} \tilde{\mathcal{R}}_{M,j} := & \prod_{i \in \mathbb{Z}/n\mathbb{Z}} s_{v_{i^+}, v_{i+1, i+1^-}} \circ \prod_{i \in \mathbb{Z}/n\mathbb{Z}} s_{v_{i, i-1^+}, v_{i^-, i-1}} \circ s_{j-2, j-1} \\ & \circ (\mu_j \circ \mu_{j+1} \cdots \circ \mu_{j-4} \circ \mu_{j-3}) \circ (\mu_{j-1} \circ \mu_{j-2} \circ \cdots \circ \mu_{j+1} \circ \mu_j). \end{aligned} \quad (3.6)$$

The operation $\tilde{\mathcal{R}}_{M,j}$ acts both on the quiver $\tilde{Q}_{n,m}$, sending it to $\tilde{\mathcal{R}}_{M,j}(\tilde{Q}_{n,m})$, and acts as a rational transformation of $\tilde{\mathcal{F}}$.

Theorem 3.3. *For the enriched quiver $\tilde{Q}_{n,m}$, let $(\mathbf{x}_{Q_{n,m}}, \mathbf{X}_{Q_{n,m}})$ be the cluster consisting of mutable variables and frozen variables. For any $j \in \mathbb{Z}/n\mathbb{Z}$, we have*

$$\tilde{\mathcal{R}}_{M,j}(\tilde{Q}_{n,m}, \mathbf{x}_{Q_{n,m}}, \mathbf{X}_{Q_{n,m}}) = (\tilde{Q}_{n,m}, \tilde{R}_M(\mathbf{x}_{Q_{n,m}}), \tilde{R}_M(\mathbf{X}_{Q_{n,m}})).$$

Theorem 3.4. *The $m-1$ transformations $\tilde{R}_{M_1}, \tilde{R}_{M_2}, \dots, \tilde{R}_{M_{m-1}}: \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}}$ satisfy $\tilde{R}_{M_i} \tilde{R}_{M_{i+1}} \tilde{R}_{M_i} = \tilde{R}_{M_{i+1}} \tilde{R}_{M_i} \tilde{R}_{M_{i+1}}$ and $\tilde{R}_{M_i}^2 = \text{id}$. Furthermore, for i, j such that $|i - j| > 1$ we have $\tilde{R}_{M_i} \tilde{R}_{M_j} = \tilde{R}_{M_j} \tilde{R}_{M_i}$. Thus \tilde{R}_{M_i} generates an action of the symmetric group \mathfrak{S}_m on $\tilde{\mathcal{F}}$.*

See §8.3 and §8.4 for the proofs.

4. GEOMETRIC R -MATRIX

4.1. Definition of the geometric R -matrix. Let $\mathbf{p} = (p_1, p_2, \dots, p_n)$ and $\mathbf{q} = (q_1, q_2, \dots, q_n)$, where indices are taken modulo n . Define the polynomials $\kappa_i(\mathbf{p}, \mathbf{q}) \in \mathbb{C}[\mathbf{p}, \mathbf{q}]$ for $i \in \mathbb{Z}/n\mathbb{Z}$ by

$$\kappa_i(\mathbf{p}, \mathbf{q}) = \sum_{j=0}^{n-1} \prod_{\ell=1}^j p_{i-\ell} \prod_{\ell=j+2}^n q_{i-\ell}.$$

Define the rational map $R: \mathbb{C}(\mathbf{p}, \mathbf{q}) \rightarrow \mathbb{C}(\mathbf{p}, \mathbf{q})$, called the *geometric R -matrix* by the formula

$$R(p_i) = q_i \frac{\kappa_{i+1}(\mathbf{p}, \mathbf{q})}{\kappa_i(\mathbf{p}, \mathbf{q})}, \quad R(q_i) = p_i \frac{\kappa_i(\mathbf{p}, \mathbf{q})}{\kappa_{i+1}(\mathbf{p}, \mathbf{q})}.$$

The geometric R -matrix appears in the theory of geometric crystals [KNO, Eti], in study of total positivity in loop groups [LP12] and in other places [KNY].

Example 4.1. For $n = 4$ we have

$$R(q_1) = p_1 \frac{q_1 q_2 q_3 + p_4 q_1 q_2 + p_3 p_4 q_1 + p_2 p_3 p_4}{q_2 q_3 q_4 + p_1 q_2 q_3 + p_4 p_1 q_2 + p_3 p_4 p_1}.$$

4.2. From the cluster R -matrix to the geometric R -matrix. We picture the $2n$ variables \mathbf{p} and \mathbf{q} as lying on a cylindrical network as indicated in Figure 4. This network consists of two parallel vertical wires going around the cylinder, and n horizontal wires going from left to right. The variables \mathbf{p} and \mathbf{q} are placed at the $2n$ vertices.

Associate a *chamber variable*, or τ -function x to each of the chambers into which the wires partition the cylinder. Index these chamber variables so that the vertex q_i is surrounded by $x_i, x_{i+1}, x_i^+, x_{i+1}^+$, while p_i is surrounded by $x_i^-, x_{i+1}^-, x_i, x_{i+1}$; see Figure 6. We may superimpose the triangular grid quiver $Q_{n,2}$ onto the network, where we identify $M_0, M_1, M_2 \subset Q_{n,2}$ with M^-, M, M^+ respectively. In the corresponding enriched quiver $\tilde{Q}_{n,2}$ remove all the frozen variables of the form $X_{i,i+1}$ and X_{i,i^-} (or equivalently, set those frozen variables to 1). Call the resulting quiver $\tilde{Q}'_{n,2}$.

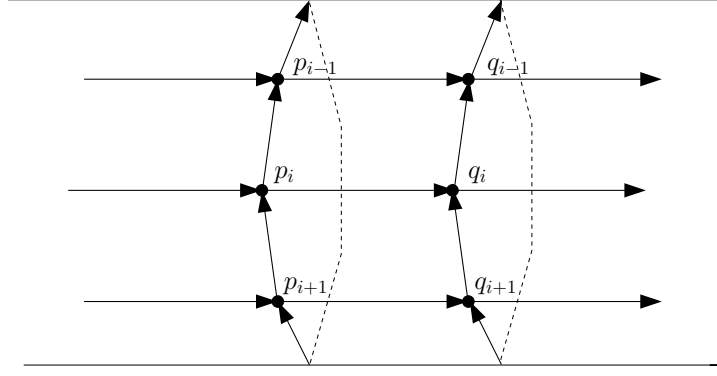


FIGURE 4. Adjacent parallel vertical wires on a cylinder.

The above procedure is a special case of a general way to associate a quiver to a wiring diagram. Specifically, for any chamber of the wiring diagram there are generically six arrows in the quiver incident to it. They alternate in direction and connect the chamber to the three “most forward” and the three “most backward” chambers neighboring it, as shown in Figure 5. In special cases, for example when the original chamber being a triangle for example, some of the six arrows cancel each other out and one is left with only four arrows. This construction is usually done for *planar* wiring diagrams,

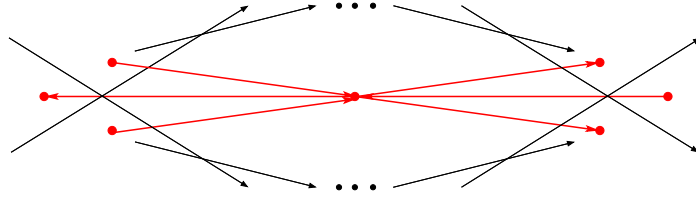


FIGURE 5. Associating a quiver to a wiring diagram.

see [BFZ05] for an equivalent rule. Our cylindric networks are locally planar and applying the construction to $N_{n,m}$, we obtain $\tilde{Q}'_{n,m}$.

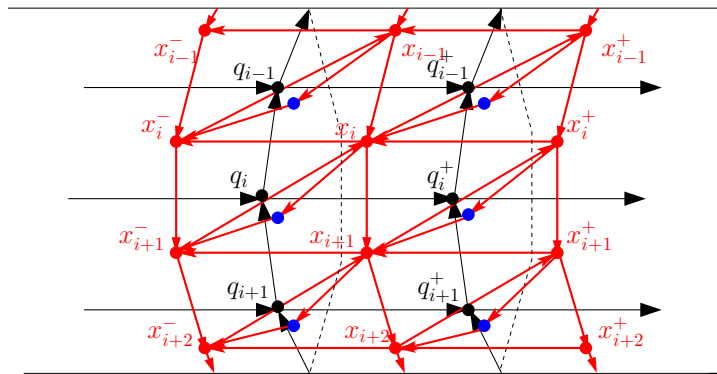


FIGURE 6. The triangular grid quiver superimposed with the cylindrical network.

Define an algebra map $\iota: \mathbb{C}(\mathbf{p}, \mathbf{q}) \rightarrow \mathcal{F}(\tilde{Q}'_{n,2}) = \mathbb{C}(\mathbf{x}^-, \mathbf{x}, \mathbf{x}^+, X_{i+1^-,i}, X_{i+1,i^+})$ by

$$\iota(p_i) = \frac{x_i^- x_{i+1}}{x_{i+1}^- x_i} X_{i+1^-,i}, \quad \iota(q_i) = \frac{x_i x_{i+1}^+}{x_{i+1} x_i^+} X_{i+1,i^+} \quad (4.1)$$

for all $i \in \mathbb{Z}/n\mathbb{Z}$. It is clear that ι is an inclusion of fields. Abusing notation, we still denote the cluster R -matrix (obtained by specializing certain frozen variables to 1 in (3.4) and (3.5)) of $\mathcal{F}(\tilde{Q}'_{n,2})$ by $\tilde{R}_\mathbf{x}$.

Theorem 4.2. *We have $\iota \circ R = \tilde{R}_\mathbf{x} \circ \iota$.*

Proof. We first compute that

$$\begin{aligned} \iota(\kappa_i(\mathbf{p}, \mathbf{q})) &= \sum_{j=0}^{n-1} \left(\prod_{\ell=1}^j \frac{x_{i-\ell}^- x_{i+1-\ell}}{x_{i+1-\ell}^- x_{i-\ell}^+} X_{i+1-\ell^-, i-\ell} \right) \left(\prod_{\ell=j+2}^n \frac{x_{i-\ell}^- x_{i+1-\ell}^+}{x_{i+1-\ell}^- x_{i-\ell}^+} X_{i+1-\ell^-, i-\ell^+} \right) \\ &= \frac{x_i^2}{x_i^- x_i^+} \sum_{j=0}^{n-1} \frac{x_{i-j}^- x_{i-j-1}^+}{x_{i-j}^- x_{i-j-1}^+} \prod_{\ell=1}^j X_{i+1-\ell^-, i-\ell} \prod_{\ell=j+2}^n X_{i+1-\ell^-, i-\ell^+}. \end{aligned}$$

Thus

$$\begin{aligned} \iota(R(p_i)) &= \iota(q_i) \frac{\iota(\kappa_{i+1}(\mathbf{p}, \mathbf{q}))}{\iota(\kappa_i(\mathbf{p}, \mathbf{q}))} \\ &= \frac{x_i^- x_{i+1}}{x_{i+1}^- x_i} X_{i+1, i^+} \frac{\sum_{j=0}^{n-1} x_{i+1-j}^- \left(\prod_{r \in \mathbb{Z}/n\mathbb{Z} \setminus \{i-j, i+1-j\}} x_i \right) x_{i-j}^+ \prod_{\ell=1}^j X_{i+2-\ell^-, i+1-\ell} \prod_{\ell=j+2}^n X_{i+2-\ell^-, i+1-\ell^+}}{\sum_{j=0}^{n-1} x_{i-j}^- \left(\prod_{r \in \mathbb{Z}/n\mathbb{Z} \setminus \{i-1-j, i-j\}} x_i \right) x_{i-j-1}^+ \prod_{\ell=1}^j X_{i+1-\ell^-, i-\ell} \prod_{\ell=j+2}^n X_{i+1-\ell^-, i-\ell^+}}. \end{aligned}$$

Re-indexing the summations, we see that this is equal to

$$\tilde{R}_\mathbf{x}(\iota(p_i)) = \frac{x_i^- x_{i+1}}{x_{i+1}^- x_i} X_{i+1, i^+} \frac{\sum_{j=1}^n x_{j+1}^- \left(\prod_{\ell=j+2}^{j-1} x_\ell \right) x_j^+ \prod_{\ell=j+1}^i X_{\ell+1^-, \ell} \prod_{\ell=i+2}^j X_{\ell, \ell-1^+}}{\sum_{j=1}^n x_{j+1}^- \left(\prod_{\ell=j+2}^{j-1} x_\ell \right) x_j^+ \prod_{\ell=j+1}^{i-1} X_{\ell+1^-, \ell} \prod_{\ell=i+1}^j X_{\ell, \ell-1^+}},$$

as required. \square

Let $N_{n,m}$ denote a network on a cylinder with n parallel horizontal wires and m parallel vertical wires W_1, W_2, \dots, W_m which go around the cylinder (oriented as in Figure 4). Each circle W_j has n crossings with the horizontal wires. Let $q_{j,i}$ be a variable assigned to the i -th crossing on W_j , where we assume $i \in \mathbb{Z}/n\mathbb{Z}$.

Let $\mathcal{Q} = \mathcal{Q}(N_{n,m}) := \mathbb{C}(\mathbf{q}_1, \dots, \mathbf{q}_m)$ where $\mathbf{q}_j = (q_{j,i})_{i=1, \dots, n}$. Define the quiver $\tilde{Q}'_{n,m}$ by removing frozen vertices as before. Define an algebra map $\iota_m: \mathcal{Q} \rightarrow \mathcal{F}(\tilde{Q}'_{n,m})$ by

$$\iota_m(q_{j,i}) = \frac{x_{j-1,i} x_{j,i+1}}{x_{j-1,i+1} x_{j,i}} X_{v_{(j-1,i+1), (j,i)}}. \quad (4.2)$$

This reminds us of the τ -function substitution, where the cluster variables play the role of τ -functions. The following result is well known [KNY, Eti, LP12]. From Theorem 3.4, we obtain a new proof via the cluster R -matrix.

Corollary 4.3. *Let R_j , for $j = 1, 2, \dots, m-1$, act on the variables \mathbf{q}_j and \mathbf{q}_{j+1} by R and fixing all the other variables in \mathcal{Q} . Then we have $R_j R_{j+1} R_j = R_{j+1} R_j R_{j+1}$ and*

$R_j^2 = \text{id}$. Furthermore, for i, j such that $|i - j| > 1$ we have $R_i R_j = R_j R_i$. Thus R_j generates an action of \mathfrak{S}_m on \mathcal{Q} .

5. QUANTUM GEOMETRIC R -MATRIX

5.1. Quantum torus. We refer the reader to [BZ, Section 4] for a more systematic treatment of quantum tori.

Let $\mathbf{p} = (p_1, p_2, \dots, p_n)$ and $\mathbf{q} = (q_1, q_2, \dots, q_n)$ where the indices are taken modulo n . Let $\mathbb{C}_\epsilon[\mathbf{p}, \mathbf{q}]$ be the noncommutative $\mathbb{C}[\epsilon]$ -algebra generated by $p_i, p_i^{-1}, q_i, q_i^{-1}$ with the commutation relations

$$p_i q_j = \begin{cases} \epsilon q_j p_i & \text{if } j = i \text{ or } j = i - 2, \\ \epsilon^{-2} q_j p_i & \text{if } j = i - 1, \\ q_j p_i & \text{otherwise.} \end{cases}$$

and for $\mathbf{r} = \mathbf{p}$ or $\mathbf{r} = \mathbf{q}$, the relations

$$r_i r_j = \begin{cases} \epsilon r_j r_i & \text{if } j = i - 1, \\ \epsilon^{-1} r_j r_i & \text{if } j = i + 1, \\ r_j r_i & \text{otherwise.} \end{cases}$$

Note that the relations are preserved by a $\mathbb{Z}/n\mathbb{Z}$ -shift of the indices. The subalgebra $\mathbb{C}[\epsilon]$ is central. The ring $\mathbb{C}_\epsilon[\mathbf{p}, \mathbf{q}]$ is a *quantum torus*; it has a distinguished $\mathbb{C}[\epsilon]$ -basis given by the monomials $p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n} q_1^{b_1} q_2^{b_2} \cdots q_n^{b_n}$, where $a_i, b_i \in \mathbb{Z}$. The quantum torus $\mathbb{C}_\epsilon[\mathbf{p}, \mathbf{q}]$ is an Ore domain [BZ, Appendix]. It includes into its skew field of fractions $\mathbb{C}_\epsilon\langle \mathbf{p}, \mathbf{q} \rangle$.

5.2. Definition of the quantum geometric R -matrix. Define the *quantum geometric R -matrix* $R^\epsilon: \mathbb{C}_\epsilon\langle \mathbf{p}, \mathbf{q} \rangle \rightarrow \mathbb{C}_\epsilon\langle \mathbf{p}, \mathbf{q} \rangle$ by

$$R^\epsilon(p_i) = (\kappa_i^\epsilon(\mathbf{p}, \mathbf{q}))^{-1} \cdot q_i \cdot \kappa_{i+1}^\epsilon(\mathbf{p}, \mathbf{q}), \quad R^\epsilon(q_i) = (\kappa_{i+1}^\epsilon(\mathbf{p}, \mathbf{q}))^{-1} \cdot p_i \cdot \kappa_i^\epsilon(\mathbf{p}, \mathbf{q}) \quad (5.1)$$

where

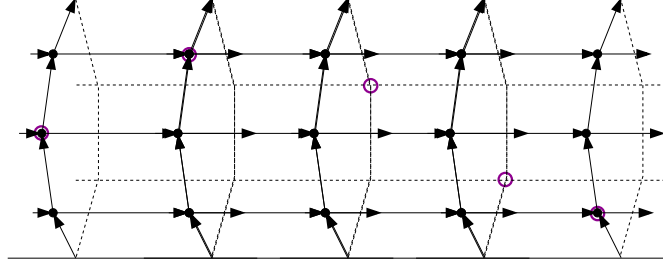
$$\kappa_i^\epsilon(\mathbf{p}, \mathbf{q}) := \sum_{j=0}^{n-1} p_{i-1} p_{i-2} \cdots p_{i-j} q_{i-j-2} q_{i-j-3} \cdots q_{i-n} \in \mathbb{C}_\epsilon\langle \mathbf{p}, \mathbf{q} \rangle, \quad (5.2)$$

for $i \in \mathbb{Z}/n\mathbb{Z}$. Though the polynomials κ_i^ϵ are apparently the same as the classical case, the variables p_i and q_i are now ϵ -commuting. It follows from Theorem 5.2 below that R^ϵ is a morphism of noncommutative $\mathbb{C}[\epsilon]$ -algebras.

5.3. Yang-Baxter relation. Recall from §4.2 the definition of the cylindrical network $N_{n,m}$. Let S_k be the k -th *snake path* in $N_{n,m}$, consisting of the variables $q_{j,i}$ satisfying $i + j \equiv 1 + k \pmod{n}$. An example of a snake path is shown in Figure 7.

Definition 5.1. Let $\mathcal{Q}_\epsilon = \mathbb{C}_\epsilon\langle \mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_m \rangle$ be the skew field of the quantum torus over $\mathbb{C}[\epsilon]$ generated by $q_{j,i}$ ($1 \leq j \leq m$, $i \in \mathbb{Z}/n\mathbb{Z}$) with the commutation relations:

$$q_{ji} q_{j'i'} = \begin{cases} \epsilon q_{j'i'} q_{ji} & q_{ji} \in S_{k+1}, q_{j'i'} \in S_k, j \leq j' \\ \epsilon^{-1} q_{j'i'} q_{ji} & q_{ji} \in S_{k+1}, q_{j'i'} \in S_k, j > j' \\ \epsilon^{-2} q_{j'i'} q_{ji} & q_{ji}, q_{j'i'} \in S_k, j < j' \\ \epsilon^2 q_{j'i'} q_{ji} & q_{ji}, q_{j'i'} \in S_k, j > j' \\ q_{j'i'} q_{ji} & \text{otherwise.} \end{cases}$$

FIGURE 7. Circled vertices form a snake path in $N_{5,m}$.

Note that the commutation relations restricted to the variables \mathbf{q}_j and \mathbf{q}_{j+1} are the same as those for \mathbf{p} and \mathbf{q} in $\mathbb{C}_\epsilon\langle\mathbf{p}, \mathbf{q}\rangle$. These relations were in part inspired by [Ber].

Let R_j^ϵ , for $j = 1, 2, \dots, m-1$, act on the variables \mathbf{q}_j and \mathbf{q}_{j+1} by R^ϵ and fixing all the other variables in \mathcal{Q}_ϵ .

Theorem 5.2. *The map $R_j^\epsilon: \mathcal{Q}_\epsilon \rightarrow \mathcal{Q}_\epsilon$ is a morphism of noncommutative $\mathbb{C}[\epsilon]$ -algebras. Thus if $R^\epsilon(\mathbf{q}_j, \mathbf{q}_{j+1}) = (\mathbf{q}'_j, \mathbf{q}'_{j+1})$ then $(\mathbf{q}_1, \dots, \mathbf{q}'_j, \mathbf{q}'_{j+1}, \dots, \mathbf{q}_m)$ satisfy identical commutation relations to $(\mathbf{q}_1, \dots, \mathbf{q}_m)$.*

Theorem 5.3. *We have $R_j^\epsilon R_{j+1}^\epsilon R_j^\epsilon = R_{j+1}^\epsilon R_j^\epsilon R_{j+1}^\epsilon$ and $(R_j^\epsilon)^2 = \text{id}$. Furthermore, for i, j such that $|i - j| > 1$ we have $R_i^\epsilon R_j^\epsilon = R_j^\epsilon R_i^\epsilon$. Thus R_j^ϵ generates an action of \mathfrak{S}_m on \mathcal{Q}_ϵ .*

See §7.5 and §7.6 for the proofs.

6. INVARIANTS OF THE QUANTUM GEOMETRIC R -MATRIX

We study the invariants of the quantum geometric R -matrix action on the network $N_{n,m}$. For this purpose, we use the notion of *highway paths* on $N_{n,m}$, as studied in [LP13], and we define quantum analogues of the loop symmetric functions of [LP12].

In this section only, we use the notation $q_j^{(i)} := q_{j,i}$ to more closely match the usual notation in the theory of (loop) symmetric functions.

6.1. Highway Measurements. Consider the class of networks which are

- directed graphs embedded on an orientable surface, where edges intersect only at vertices;
- have boundary vertices of degree 1, either sources or sinks depending on the direction of the adjacent edge;
- have internal vertices of degree 4, with exactly 2 incoming edges and 2 outgoing edges, and incoming and outgoing edges not interlacing;
- have a weight assigned to each vertex, and the weights do not necessarily commute.

The cylindric network $N_{n,m}$ (see §4) with weights $q_j^{(i)}$ is an example of such a network. Given such a network N , we consider *highway measurements* $M(N)$ in these networks defined as follows. We fix ordered collections of source boundary vertices s_1, s_2, \dots, s_r and sink boundary vertices t_1, t_2, \dots, t_r of equal cardinality, and for each $i = 1, 2, \dots, r$, we fix a homology class $[\gamma_i]$ of a path from s_i to t_i . We then take the weight generating

function

$$M(N) = M_{s,t,[\gamma]}(N) := \sum_{\mathbb{P}} \text{wt}(\mathbb{P})$$

of families of paths $\mathbb{P} = (P_1, P_2, \dots, P_r)$, where P_i goes from s_i to t_i , has homology class $[P_i] = [\gamma_i]$, and the paths are required to be *non-intersecting*, that is, they may not share common edges. The weight assigned to each path P is the product of the

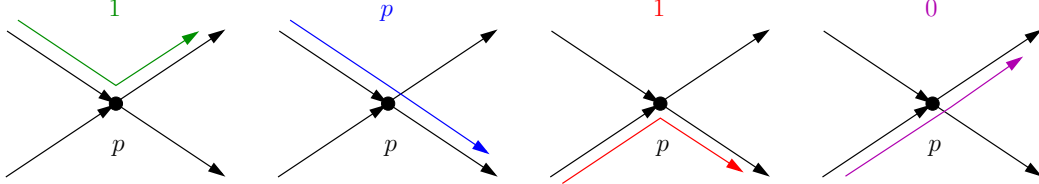


FIGURE 8. The four ways a path can pass through a vertex of a network, and the corresponding weight it picks up.

weights picked up at each of its vertices, depending on one of the four ways the path passes through the vertex, as shown in Figure 8. For each path P we take the product in the order the vertices are visited to obtain the weight $\text{wt}(P)$. The weight of a family $\mathbb{P} = (P_1, P_2, \dots, P_r)$ of paths is the product $\text{wt}(\mathbb{P}) = \text{wt}(P_1)\text{wt}(P_2) \cdots \text{wt}(P_r)$. Note that the ordering of the paths is important since we allow weights to be noncommutative. We call a path P a *highway path* if it has nonzero weight.

In §6.2, we consider path families on $N_{n,m}$ consisting of a single path. In §6.4, we consider path families on the universal cover $\tilde{N}_{n,m}$. In §6.5, we consider arbitrary path families on $N_{n,m}$. In the latter cases, we will impose conditions on the ordering of paths in path families.

6.2. Quantum loop elementary symmetric functions. Let \mathcal{Q}_ϵ be the skew field defined in Definition 5.1, with distinguished noncommuting generators now denoted $q_j^{(i)}$. Denote by $\mathcal{Q}_\epsilon^{\mathfrak{S}_m}$ the invariants of the quantum R -matrix action on \mathcal{Q}_ϵ from Theorem 5.3.

For $k = 1, 2, \dots, m$ and $r \in \mathbb{Z}/n\mathbb{Z}$, define the quantum loop elementary symmetric function

$$e_k^{(r)}(\mathbf{q}_1, \dots, \mathbf{q}_m) := \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq m} q_{j_1}^{(r+1-j_1)} q_{j_2}^{(r+2-j_2)} \cdots q_{j_k}^{(r+k-j_k)} \in \mathcal{Q}_\epsilon.$$

For example, with $n = 3$ and $m = 4$, we have

$$\begin{aligned} e_1^{(1)} &= q_1^{(1)} + q_2^{(3)} + q_3^{(2)} + q_4^{(1)} \\ e_2^{(1)} &= q_1^{(1)} q_2^{(1)} + q_1^{(1)} q_3^{(3)} + q_1^{(1)} q_4^{(2)} + q_2^{(3)} q_3^{(3)} + q_2^{(3)} q_4^{(2)} + q_3^{(2)} q_4^{(2)} \\ e_3^{(1)} &= q_1^{(1)} q_2^{(1)} q_3^{(1)} + q_1^{(1)} q_2^{(1)} q_4^{(3)} + q_1^{(1)} q_3^{(3)} q_4^{(3)} + q_2^{(3)} q_3^{(3)} q_4^{(3)} \\ e_4^{(1)} &= q_1^{(1)} q_2^{(1)} q_3^{(1)} q_4^{(1)}. \end{aligned}$$

By convention, we have $e_0^{(r)} = 1$ and $e_s^{(r)} = 0$ for $s < 0$ or $s > m$. The quantum loop elementary symmetric function $e_k^{(r)}$ can be interpreted as a highway measurement: it is the weight generating function of paths in $N_{n,m}$ with a fixed sink and fixed source. We shall show that $e_k^{(r)} \in \mathcal{Q}_\epsilon^{\mathfrak{S}_m}$.

For two matrices $A = (a_{ij})$ and $B = (b_{ij})$ whose entries lie in a possibly noncommutative ring, we define the product AB by $(AB)_{ij} = \sum_k a_{ik}b_{kj}$, where the ordering of the factors in the product $a_{ik}b_{kj}$ must be kept.

Let

$$M(\mathbf{q}; t) = \begin{pmatrix} q_1 & 0 & 0 & t \\ 1 & q_2 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & q_n \end{pmatrix},$$

where t is a spectral parameter. Let $\mathcal{M}(t) := M(\mathbf{q}_1; t)M(\mathbf{q}_2; t) \cdots M(\mathbf{q}_m; t)$. The following observation follows from the definitions.

Lemma 6.1. *We have*

$$\mathcal{M}_{i,j}(t) = \sum_{s \geq 0} e_{j-i+m-sn}^{(i)} t^s.$$

Write $(\mathbf{p}', \mathbf{q}') := R^\epsilon(\mathbf{p}, \mathbf{q})$.

Lemma 6.2. *We have $M(\mathbf{p}; t)M(\mathbf{q}; t) = M(\mathbf{p}'; t)M(\mathbf{q}'; t)$.*

Proof. We check that $p_i q_i = p'_i q'_i$ and $p_{i+1} + q_i = p'_{i+1} + q'_i$ follow from (5.1). \square

Corollary 6.3. *For $k > 0$ and $r \in \mathbb{Z}/m\mathbb{Z}$, we have $e_k^{(r)}(\mathbf{q}_1, \dots, \mathbf{q}_m) \in \mathcal{Q}_e^{\mathfrak{S}_m}$.*

Proof. By Lemma 6.2, the entries of $\mathcal{M}(t)$ belong to $\mathcal{Q}_e^{\mathfrak{S}_m}$. The result follows from Lemma 6.1. \square

We remark that the classical ($\epsilon = 1$) case of $e_k^{(r)}$ was essentially studied in [Yam, Section 2].

6.3. Yang-Baxter move. We now consider certain *local moves* that do not change the highway measurements. The most important one is the *Yang-Baxter move* shown in Figure 9.

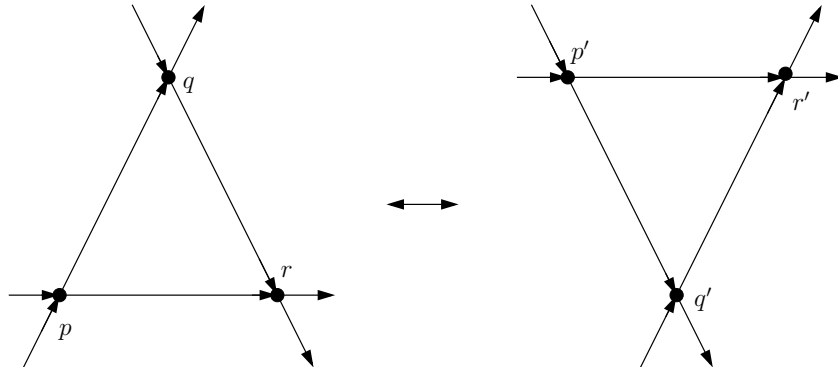


FIGURE 9. The Yang-Baxter move.

Assume p, q, r are noncommutative parameters which nevertheless satisfy the relation

$$pqr = rqp.$$

Define the transformation of parameters in the Yang-Baxter move as follows:

$$\begin{cases} p' = qr(p+r)^{-1} = (p+r)^{-1}rq, \\ q' = p+r, \\ r' = qp(p+r)^{-1} = (p+r)^{-1}pq. \end{cases} \quad (6.1)$$

In the classical $\epsilon = 1$ case, this Yang-Baxter move was studied in [LP13]. In our setting it can be compared to [Ber].

Proposition 6.4.

- (1) *The Yang-Baxter transformation preserves highway measurements locally, that is, all highway measurements of the two networks shown in Figure 9 are identical.*
- (2) *The resulting parameters again satisfy $p'q'r' = r'q'p'$.*
- (3) *The Yang-Baxter transformation is an involution.*

Proof. The only non-trivial part to verify is that the highway measurements are preserved locally. The four essentially different ways to take highway measurements are shown in Figure 10. This results in the following system of equations:

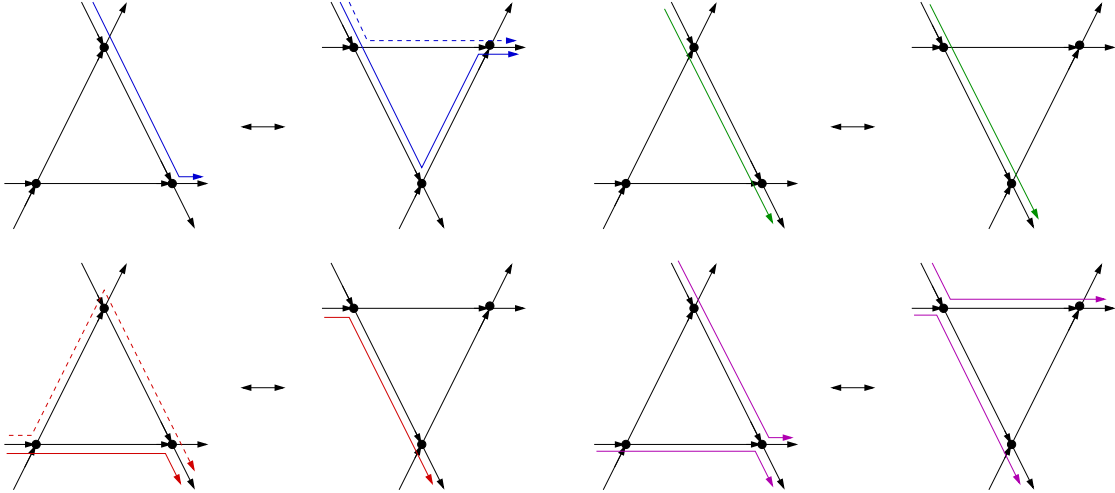


FIGURE 10. Non-trivial highway measurements in the Yang-Baxter move.

$$\begin{cases} qr = p'q', \\ p+r = q', \\ q = p' + r', \\ qp = r'q', \\ pq = q'r'. \end{cases}$$

Note that we include both equations $qp = r'q'$ and $pq = q'r'$ since we want the measurements to be conserved no matter which of the two orders we choose on the two distinct paths we have in this case. It is easy to check now that the transformation above satisfies all those relations. \square

The other local move we want is the *lens creation-annihilation* move.

Proposition 6.5. *The lens creation-annihilation transformation preserves highway measurements.*

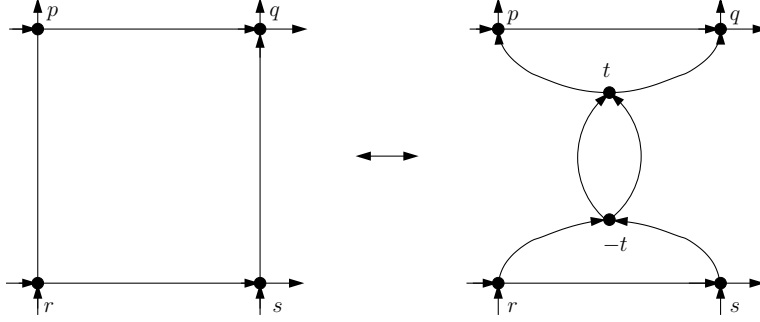


FIGURE 11. The lens creation-annihilation move.

Proof. The proof is the same as for the Yang-Baxter move. In fact, in this case the measurements with more than one path do not impose any extra relations, and thus not only local but also global measurements are preserved. In other words, if we perform the lens creation-annihilation transformation locally in a big network N , all highway measurements of N are preserved. \square

6.4. Quantum loop Schur functions. Let λ/μ be a skew shape. Let T be a semistandard Young tableau of shape λ/μ and $r \in \mathbb{Z}/n\mathbb{Z}$. Define the content $c(b)$ of a box b in the i -th row and j -th column to be $c(b) = i - j$. The r -reading word of T is the monomial $\text{wt}_\epsilon^{(r)}(T) = \prod_{b \in \lambda/\mu} q_{T(b)}^{(c(b) - T(b) + r + 1)} \in \mathcal{Q}_\epsilon$, where the product is taken by reading the columns from the left to the right, and each column is read from the top to the bottom. For example, with $n = 3$,

$$\text{for } T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 3 & \\ \hline 4 & & & \\ \hline \end{array} \quad \text{we have} \quad \text{wt}_\epsilon^{(1)} = (q_1^{(1)} q_2^{(1)} q_4^{(3)}) (q_1^{(3)} q_3^{(2)}) (q_2^{(1)} q_3^{(1)}) (q_4^{(1)}).$$

Define the *quantum loop Schur function* by

$$s_{\lambda/\mu}^{(r)}(\mathbf{q}_1, \dots, \mathbf{q}_m) = \sum_T \text{wt}_\epsilon^{(r)}(T) \in \mathcal{Q}_\epsilon$$

where the summation is over all semistandard Young tableaux T of shape λ/μ filled with the integers $1, 2, \dots, m$. Up to a shift in the upper index, our quantum loop Schur functions reduce to the loop Schur functions of [LP12] at $\epsilon = 1$. When λ/μ is a column, the quantum loop Schur function is simply the quantum loop elementary symmetric function.

Theorem 6.6. *For any skew shape λ/μ and $r \in \mathbb{Z}/n\mathbb{Z}$, we have $s_{\lambda/\mu}(\mathbf{q}_1, \dots, \mathbf{q}_m) \in \mathcal{Q}_\epsilon^{\mathfrak{S}_m}$.*

We remark that $s_{\lambda/\mu}(\mathbf{q}_1, \dots, \mathbf{q}_m)$ is not a highway measurement of $N_{n,m}$ but of the universal cover $\tilde{N}_{n,m}$. This is because it corresponds to families of paths that do not intersect on $\tilde{N}_{n,m}$, but whose images in $N_{n,m}$ may intersect.

Example 6.7. Set $n = m = 3$, $\lambda = (2, 1)$ and $\mu = \emptyset$. The eight semistandard tableaux of shape λ are

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array}. \quad (6.2)$$

Thus we have

$$s_{\lambda}^{(1)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = q_1^{(1)} q_2^{(1)} (q_1^{(3)} + q_2^{(2)} + q_3^{(1)}) + q_1^{(1)} q_3^{(3)} (q_1^{(3)} + q_2^{(2)} + q_3^{(1)}) + q_2^{(3)} q_3^{(3)} (q_2^{(2)} + q_3^{(1)}),$$

where each weight in $s_{\lambda}^{(1)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ corresponds to a pair of paths in $\tilde{N}_{3,3}$. Note that the pair of paths with weight $q_1^{(1)} q_2^{(1)} q_3^{(1)}$ intersect in $N_{3,3}$.

This section will be devoted to the proof of Theorem 6.6. We follow closely the ideas of [LP13], but in the noncommutative setting. Our treatment is briefer than the treatment in [LP13, Section 6]. The idea is to realize the quantum geometric R -matrix by

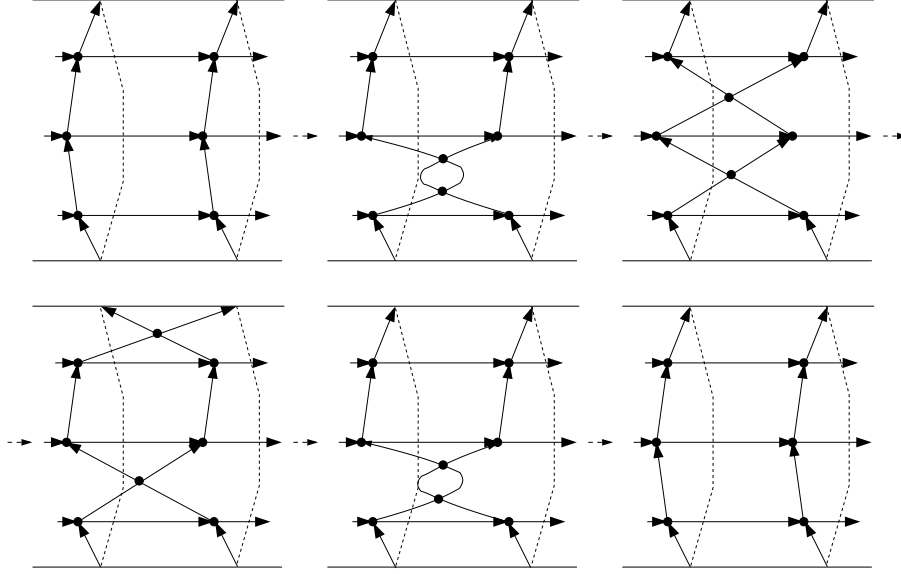


FIGURE 12. Realization of quantum geometric R -matrix via local moves.

a sequence of local moves of $N_{n,m}$, each of which preserves highway measurements. We pick two adjacent vertical wires W_j and W_{j+1} and create a lens between them. We push one of the two vertices of the lens around the cylinder through the horizontal wires using the Yang-Baxter move, and then annihilate the lens. We choose the vertex parameter of the original lens so that it comes out on the other side equal to its original value, thus allowing the annihilation to happen. The sequence of local moves is illustrated in Figure 12.

Recall the definition of $\kappa_i^\epsilon(\mathbf{p}, \mathbf{q})$ (5.2). Assume that the lens crossing being pushed is currently in the region with parameters $p_i, p_{i+1}, q_i, q_{i+1}$, and its value is equal to

$$r_{i+1} = \left(\prod_{i=n}^1 p_i - \prod_{i=n}^1 q_i \right) (\kappa_{i+1}^\epsilon)^{-1} = (\kappa_{i+1}^\epsilon)^{-1} \left(\prod_{i=n}^1 p_i - \prod_{i=n}^1 q_i \right). \quad (6.3)$$

The last equality holds since $\prod_{i=n}^1 p_i$ and $\prod_{i=n}^1 q_i$ are central elements of $\mathbb{C}_\epsilon\langle \mathbf{p}, \mathbf{q} \rangle$.

Let $\tilde{N}_{n,m}$ denote the universal cover of $N_{n,m}$. It is a network embedded into a vertical strip. We only consider highway measurements of $\tilde{N}_{n,m}$ where in a non-intersecting family $\mathbb{P} = (P_1, P_2, \dots, P_r)$, the paths P_i are ordered from bottom to top.

Proposition 6.8. *With the above notation,*

- (1) *the parameters satisfy the relation $r_{i+1} p_i q_i = q_i p_i r_{i+1}$;*

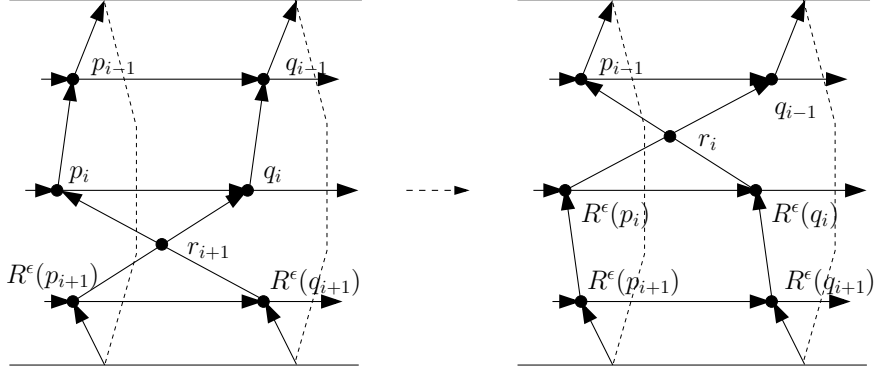


FIGURE 13. A local move.

- (2) applying the Yang-Baxter move to the parameters r_{i+1}, p_i, q_i , one obtains the parameters $R^\epsilon(p_i), R^\epsilon(q_i), r_i$;
- (3) the Yang-Baxter move preserves all highway measurements on the universal cover $\tilde{N}_{n,m}$.

Proof. First, $p_i q_i = \epsilon q_i p_i$. Next, $(\prod_{j=i-1}^{i+1} q_i) q_i = q_i (\prod_{j=i-1}^{i+1} q_i)$ and $(\prod_{j=i-1}^{i+1} q_i) p_i = \epsilon^{-1} p_i (\prod_{j=i-1}^{i+1} q_i)$. All other terms in κ_{i+1} satisfy the same commutation relations with p_i and q_i , giving

$$(\kappa_{i+1}^\epsilon - \prod_{j=i-1}^{i+1} q_i) q_i = \epsilon^{-2} q_i (\kappa_{i+1}^\epsilon - \prod_{j=i-1}^{i+1} q_i) \quad \text{and} \quad (\kappa_{i+1}^\epsilon - \prod_{j=i-1}^{i+1} q_i) p_i = \epsilon p_i (\kappa_{i+1}^\epsilon - \prod_{j=i-1}^{i+1} q_i).$$

Combining, we see that

$$p_i q_i (\kappa_{i+1}^\epsilon) = \epsilon^{-1} (\kappa_{i+1}^\epsilon) p_i q_i. \quad (6.4)$$

Using $p_i q_i = \epsilon q_i p_i$, we obtain (1).

For the second claim, we apply the formulae (6.1) of the Yang-Baxter move. We have

$$\begin{aligned} (r_{i+1} + q_i)^{-1} r_{i+1} p_i &= \left[(\kappa_{i+1}^\epsilon)^{-1} \left(\prod_j p_j - \prod_j q_j + \kappa_{i+1}^\epsilon q_i \right) \right]^{-1} (\kappa_{i+1}^\epsilon)^{-1} \left(\prod_j p_j - \prod_j q_j \right) p_i \\ &= \left[\prod_j p_j - \prod_j q_j + \kappa_{i+1}^\epsilon q_i \right]^{-1} p_i \left(\prod_j p_j - \prod_j q_j \right) = [p_i \kappa_i^\epsilon]^{-1} p_i \left(\prod_j p_j - \prod_j q_j \right) = r_i. \end{aligned}$$

Also, in the process we saw that

$$r_{i+1} + q_i = (\kappa_{i+1}^\epsilon)^{-1} p_i \kappa_i^\epsilon = R^\epsilon(q_i),$$

and the argument for $R^\epsilon(p_i)$ is similar. This proves (2).

Finally, we argue that these Yang-Baxter moves preserve all highway measurements of $N_{n,m}$. The only non-trivial case is the case when there are two paths running through the parameters of the Yang-Baxter move, both picking up a non-trivial weight. The reason this case is non-trivial is as follows: while we know from Proposition 6.4 that highway measurements are preserved locally, in the highway measurement of $N_{n,m}$ those two parameters are separated by several other (noncommuting) factors, since they belong to different paths. We call these factors *the middle factors*. This case is shown in Figure 14.

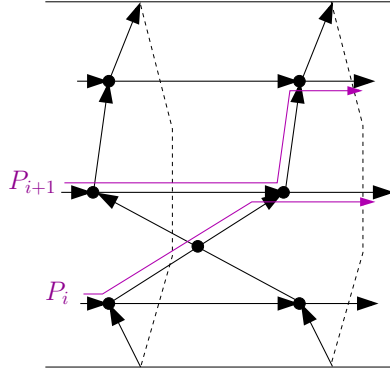


FIGURE 14. The non-trivial case to consider for preservation of global measurements.

Since we assumed that paths in a non-intersecting family are ordered from bottom to top, the two highway paths involved in Figure 14 must be P_i and P_{i+1} for some i . In this case the weight of the family of highway paths has the following form:

$$\text{wt}_\epsilon(P_1)\text{wt}_\epsilon(P_2)\cdots\text{wt}_\epsilon(P_{i-1})\cdots r_{i+1}(\text{ middle factors })p_i\cdots\text{wt}_\epsilon(P_{i+2})\cdots\text{wt}_\epsilon(P_r).$$

By Proposition 6.4(1), we have $r_{i+1}p_i = R^\epsilon(q_i)r_i$ and $p_i r_{i+1} = r_i R^\epsilon(q_i)$. We still need to commute r_{i+1} or p_i through the middle factors in order to be able to apply these relations, and then we need to commute the result back. However, we claim the following: r_{i+1} and $R^\epsilon(q_i)$ satisfy the same commutation relations with the middle factors as p_{i+1} and q_i do. Indeed, it is clear from Figure 14 that all the middle factors come from vertices that are strictly to the left or strictly to the right of the p -s and the q -s. This means the commutation relations are determined by the snake paths on which the variables lie. To see then that $(\kappa_{i+1}^\epsilon)^{-1}$ satisfies the same relation as p_{i+1} and q_i is a matter of a simple verification, which can be shortened by remembering that $\prod_i p_i$ and $\prod_i q_i$ are central, and that terms in κ_{i+1}^ϵ differ from these products by omitting either p_{i+1} or q_i . \square

Corollary 6.9. *The sequence of local moves in Figure 12, with lens vertex weight given by (6.3), acts on the vertex weights of $N_{n,m}$ by the quantum geometric R -matrix.*

Proof. Since $r_{i+n} = r_i$ in (6.3), when the parameter is pushed through all n wires, it acquires the original value, and thus one can carry out the lens annihilation move. Thus by Proposition 6.8(2), the sequence local moves in Figure 12 indeed realizes the quantum geometric R -matrix. \square

Proof of Theorem 6.6. It follows from the definition of $\text{wt}_\epsilon^{(r)}(T)$ that quantum loop Schur functions are highway measurements of the universal cover $\tilde{N}_{n,m}$ of the network $N_{n,m}$. Namely, we consider families of non-intersecting paths on $\tilde{N}_{n,m}$ with fixed sources and sinks, with the paths ordered from bottom to top. Corollary 6.9 applies also to $\tilde{N}_{n,m}$. By Proposition 6.8(3), it follows that the quantum loop Schur function is preserved by the quantum geometric R -matrix. \square

6.5. Quantum cylindric loop Schur functions. To avoid certain degenerate situations, in this section we assume that $n > 2$. Cylindric loop Schur functions were defined in [LPS] where they were shown to be invariants of the geometric R -matrix. We define their quantum analogues.

Fix an integer s . Let \mathbb{Z}^2 denote the integer lattice with points (x, y) . Define the cylinder \mathcal{C}_s to be the following quotient of integer lattice:

$$\mathcal{C}_s = \mathbb{Z}^2 / (n - s, s)\mathbb{Z}.$$

In other words, \mathcal{C}_s is the quotient of \mathbb{Z}^2 by the shift that sends (x, y) into $(x + n - s, y + s)$. The set \mathcal{C}_s inherits a natural partial order from that on \mathbb{Z}^2 given by the transitive closure of the cover relations $(x, y) < (x + 1, b)$ and $(x, y) < (x, y - 1)$. A box in the i -th row and j -th column of a Young diagram has coordinates $(j, -i)$.

A *cylindric skew shape* D is a finite convex subposet of \mathcal{C}_s . A semistandard Young tableau of a cylindric skew shape D is a map $T: D \rightarrow \mathbb{Z}_{>0}$ satisfying $T(x, y) \leq T(x + 1, y)$ and $T(x, y) < T(x, y - 1)$ whenever the corresponding boxes lie in D .

Let D be a cylindric skew shape and $r \in \mathbb{Z}/n\mathbb{Z}$. Viewing D as an infinite periodic skew shape in \mathbb{Z}^2 , fix a fundamental domain consisting of $n - s$ consecutive columns of D , and order these columns from left to right. For a cylindric semistandard Young tableau T of shape D , the r -reading word of T is the monomial $\text{wt}_\epsilon^{(r)}(T) = \prod_{b \in D} q_{T(b)}^{(c(b) - T(b) + r + 1)} \in \mathcal{Q}_\epsilon$, where the product is taken by reading the columns in the chosen order, and each column is read from the top to the bottom.

Example 6.10. Set $n = 4$, $s = 2$, and consider the shape D in $\mathcal{C}_2 = \mathbb{Z}^2 / (2, 2)\mathbb{Z}$, whose restriction to two adjacent columns is given by the shape $\lambda = (2, 1, 1)$. For example, a cylindric semistandard Young tableau T of shape D is given by extending

$$T = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array}$$

periodically, and we have $\text{wt}_\epsilon^{(1)}(T) = (q_1^{(1)} q_2^{(1)} q_4^{(4)})(q_4^{(1)})$. Note that some semistandard Young tableaux of shape λ may not give cylindric semistandard Young tableaux, for example,

$$\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array}.$$

For a cylindric skew shape D , we define the *quantum cylindric loop Schur function* by

$$s_D^{(r)}(\mathbf{q}_1, \dots, \mathbf{q}_m) = \sum_T x^{\text{wt}_\epsilon^{(r)}(T)} \in \mathcal{Q}_\epsilon,$$

where the summation is over all semistandard Young tableaux of cylindric skew shape D filled with the integers $1, 2, \dots, m$. Up to a shift in the upper index, our quantum cylindric loop Schur functions reduce to the cylindric loop Schur functions of [LPS] at $\epsilon = 1$.

Theorem 6.11. *For any cylindric skew shape D and $r \in \mathbb{Z}/n\mathbb{Z}$, we have $s_D^{(r)}(\mathbf{q}_1, \dots, \mathbf{q}_m) \in \mathcal{Q}_\epsilon^{\mathfrak{S}_m}$.*

Example 6.12. Set $n = m = 3$, $s = 1$, and consider the shape D in $\mathcal{C}_2 = \mathbb{Z}^2 / (2, 1)\mathbb{Z}$ whose restriction to two adjacent columns is given by the shape $\lambda = (2, 1)$. The seven

possible cylindric semistandard tableaux of shape D are given by extending the tableaux from (6.2) periodically, except for the third tableau in that list. Then we have

$$s_D^{(1)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = q_1^{(1)} q_2^{(1)} (q_1^{(3)} + q_2^{(2)}) + q_1^{(1)} q_3^{(3)} (q_1^{(3)} + q_2^{(2)} + q_3^{(1)}) + q_2^{(3)} q_3^{(3)} (q_2^{(2)} + q_3^{(1)}).$$

The rest of this subsection is devoted to the proof of Theorem 6.11. The idea is to write $s_D^{(r)}(\mathbf{q}_1, \dots, \mathbf{q}_m)$ as a polynomial in the quantum loop elementary symmetric functions $e_k^{(r)} \in \mathcal{Q}_\epsilon$. There is some similarity to the proof of [LP12, Theorem 3.5].

For a highway path P in $N_{n,m}$ we abuse notation by also writing P for its weight $\text{wt}(P)$. If $\mathbb{P} = (P_1, P_2, \dots, P_k)$ is an ordered family of (possibly intersecting) highway paths on $N_{n,m}$, its weight $\text{wt}(\mathbb{P})$ is the concatenation $P_1 P_2 \cdots P_k$.

For two monomials $m, m' \in \mathcal{Q}_\epsilon$, let us define $\alpha(m, m')$ by $mm' = \epsilon^{\alpha(m, m')} m' m$. Note that $\alpha(m', m) = -\alpha(m, m')$.

Let $\tilde{N}_{n,m}$ denote the universal cover of $N_{n,m}$, where the vertices have weights $q_j^{(i)}$ arranged periodically. Note that the snake paths (see §5.3) of $\tilde{N}_{n,m}$ are naturally indexed by \mathbb{Z} . For a highway path P on the universal cover $\tilde{N}_{n,m}$, we write $s(P) \subset \mathbb{Z}$ for the set of snake paths in $\tilde{N}_{n,m}$ that make a contribution to the weight $\text{wt}(P)$. For a highway path P such that $s(P) \neq \emptyset$, $s(P)$ is a set of consecutive numbers. When $s(P) = [a, b]$, the path P picks up one q_{ji} on each snake path S_k for $k \in [a, b]$, and the monomial P has degree $b - a + 1$.

Define a function χ on \mathbb{Z} by

$$\chi(z) = \chi_n(z) = \begin{cases} 1 & \text{if } z \equiv 0 \pmod{n}, \\ 0 & \text{otherwise.} \end{cases}$$

For a path P and the weight q of an interior vertex of $\tilde{N}_{n,m}$, we write $q \in P$ if P picks up q . Note that the weights of $\tilde{N}_{n,m}$ are now $q_j^{(i)}$ where $i \in \mathbb{Z}$, but the commutation relations for $q_j^{(i)}$ only depends on $i \in \mathbb{Z}/n\mathbb{Z}$.

Lemma 6.13. *Let $P = P^{(1)} P^{(2)}$ be a decomposition of a highway path, where $s(P^{(1)}) = [a, b]$ and $s(P^{(2)}) = [b+1, c]$. Then $\alpha(P^{(1)}, P^{(2)}) = 1 - \chi(a-b-1) - \chi(b-c) + \chi(a-c-1)$.*

Proof. Write $P^{(1)} = q_a q_{a+1} \cdots q_b$ and $P^{(2)} = q_{b+1} q_{b+2} \cdots q_c$, where q_i is the weight of a vertex belonging to the i -th snake path. For $q_i \in P^{(1)}$ and $q_j \in P^{(2)}$ we have

$$\alpha(q_i, q_j) = \chi(i+1-j) - 2\chi(i-j) + \chi(i-1-j), \quad (6.5)$$

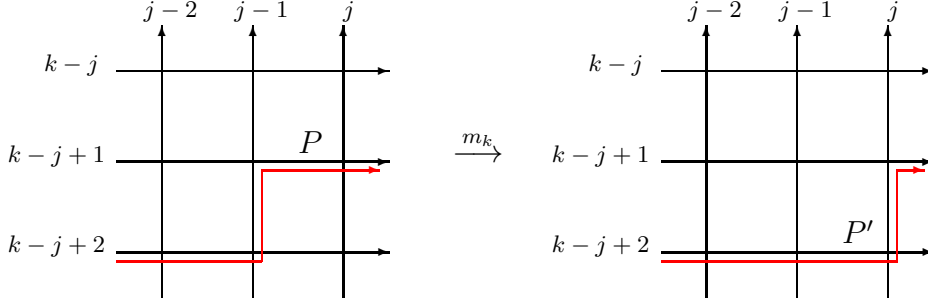
since on the cylinder the vertex with weight q_j is to the right of the vertex with weight q_i . Therefore, $\alpha(P^{(1)}, P^{(2)})$ is calculated by summing up $\alpha(q_i, q_j)$:

$$\begin{aligned} \alpha(P^{(1)}, P^{(2)}) &= \sum_{i=a}^b \sum_{j=b+1}^c \alpha(q_i, q_j) \\ &= \chi(0) - \chi(a-b-1) - \chi(b-c) + \chi(a-c-1). \end{aligned}$$

□

We write (j, i) for the intersection of the i -th horizontal wire and the j -th vertical wire in $\tilde{N}_{n,m}$. For $k \in \mathbb{Z}$ we define a map m_k on the highway paths, which locally changes a highway path P to P' in the following way. If $k \in s(P)$ and if there exists $j > 1$ such that $q_j^{(k+1-j)} \in P$ and $q_{j-1}^{(k+2-j)} \notin P$, remove the two edges $(j-1, k +$

$2-j) \rightarrow (j-1, k+1-j) \rightarrow (j, k+1-j)$ from the path P and add the two edges $(j-1, k+2-j) \rightarrow (j, k+2-j) \rightarrow (j, k+1-j)$. Otherwise, m_k does not change P . Consequently we obtain P' by replacing $q_j^{(k+1-j)}$ with $q_{j-1}^{(k+2-j)}$ in P . We remark that $q_j^{(k+1-j)}, q_{j-1}^{(k+2-j)} \in S_k$.

FIGURE 15. Action of m_k .

Lemma 6.14. *Let P and Q be highway paths on $\tilde{N}_{n,m}$ that do not intersect on $N_{n,m}$, with $s(P) = [a, b]$ and $s(Q) = [c, d]$. Assume that P and Q have distinct endpoints on the same vertical line, but that Q starts to the left of P . Also assume that $c < a$. Then, the following statements hold.*

- (1) *If P and $m_k(Q)$ do not intersect on $N_{n,m}$, then $\alpha(P, Q) = \alpha(P, m_k(Q))$.*
- (2) *If $m_k(P)$ and Q do not intersect on $N_{n,m}$, then $\alpha(P, Q) = \alpha(m_k(P), Q)$.*

Proof. By translating Q up or down tn positions for some $t \in \mathbb{Z}$, we may assume that Q lies above P but is never more than n positions above P . In the same manner as in the proof of Lemma 6.13, we write $P = p_a p_{a+1} \cdots p_b$ and $Q = q_c q_{c+1} \cdots q_d$. For the weight $q_j^{(i)}$ at a crossing (j, i) , we write $h(q_j^{(i)}) = j$ for the horizontal position.

We prove (1). Assume that $k \in [c, d]$, and let $Q' = m_k(Q)$. Then $h(q'_k) = h(q_k) - 1$. Note that $\alpha(p_i, q_k) \neq 0$ holds only when $i \equiv k, k \pm 1 \pmod{n}$. We need to check that $\alpha(p_i, q_k) = \alpha(p_i, q'_k)$ in these cases.

When $i \equiv k \pmod{n}$, as P does not intersect Q and Q' on $N_{n,m}$, we have the following:

- if $k \in [a, b]$, then $h(p_k) < h(q_k) - 1$,
- for all $j > 0$ such that $k - jn \in [a, b]$, we have $h(p_{k-jn}) < h(q_k) - n - 1$,
- for all $j > 0$ such that $k + jn \in [a, b]$, we have $h(p_{k+jn}) > h(q_k)$,

From the first property, it follows that $h(p_k) < h(q'_k)$, and we obtain $\alpha(p_k, q_k) = \alpha(p_k, q'_k) = -2$. From the second and third properties, it follows that $h(p_{k-jn}) < h(q'_k) - n$ or $h(p_{k+jn}) > h(q'_k)$, and we obtain $\alpha(p_{k \mp jn}, q_k) = \alpha(p_{k \mp jn}, q'_k) = \mp 2$.

When $i \equiv k \pm 1 \pmod{n}$, in the same manner we obtain $\alpha(p_i, q_k) = \alpha(p_i, q'_k)$ from

- if $k+1 \in [a, b]$, then $h(p_{k+1}) < h(q_k)$,
- for all $j > 0$ such that $k+1-jn \in [a, b]$, then $h(p_{k+1-jn}) < h(q_k) - n$,
- for all $j > 0$ such that $k+1+jn \in [a, b]$, then $h(p_{k+1+jn}) > h(q_k)$,
- if $k-1 \in [a, b]$, then $h(p_{k-1}) < h(q_k) - 2$,
- for all $j > 0$ such that $k-1-jn \in [a, b]$, then $h(p_{k-1-jn}) < h(q_k) - n - 2$,

- for all $j > 0$ such that $k - 1 + jn \in [a, b]$, then $h(p_{k-1+jn}) > h(q_k)$.

The proof of (2) is similar. \square

Lemma 6.15. *Let P and Q be highway paths whose images in $N_{n,m}$ are non-intersecting, with $s(P) = [a, b]$ and $s(Q) = [c, d]$. Assume that P and Q have distinct endpoints on the same vertical line, but that Q starts to the left of P . Assume that $c < a$. Then we have $\alpha(P, Q) = -\chi(c - b - 1) + \chi(c - a) + \chi(d - a + 1)$.*

Proof. By applying the m_k for some $k \in [a, b] \cup [c, d]$, and using Lemma 6.14, we may assume that the paths P and Q are

$$P = q_{a-c}^{(c+1)} q_{a-c+1}^{(c+1)} \cdots q_{b-c+1}^{(c+1)},$$

$$Q = q_1^{(c)} q_2^{(c)} \cdots q_{d-c+1}^{(c)}.$$

We write $c(i, j)$ for the right-hand side of (6.5). For $q_{i-c}^{(c+1)}$ ($i \in [a, b]$) and $q_{j-c+1}^{(c)}$ ($j \in [c, d]$) we have

$$\alpha(q_{i-c}^{(c+1)}, q_{j-c+1}^{(c)}) = \begin{cases} c(i, j) & j \in [i - 1, d], \\ -c(i, j) & j \in [c, i - 2], \end{cases}$$

and $\alpha(P, Q)$ is calculated as follows:

$$\begin{aligned} \alpha(P, Q) &= \sum_{i=a}^b \left(\sum_{j=i-1}^d c(i, j) - \sum_{j=c}^{i-2} c(i, j) \right) \\ &= \chi(a - d - 1) - \chi(b - d) + \chi(a - c) - \chi(b - c + 1) + 2\chi(2) - 2\chi(1) \\ &= \chi(a - d - 1) + \chi(a - c) - \chi(b - c + 1). \end{aligned} \quad \square$$

Proof of Theorem 6.11. We first note that quantum cylindric loop Schur functions $s_D^{(\ell)}$ are exactly the highway measurements of $N_{n,m}$ (see [LPS] for more details). Let us fix source vertices $s_1 > s_2 > \cdots > s_r > s_1 - n$ and sink vertices $t_1 > t_2 > \cdots > t_1 - n$ on $\tilde{N}_{n,m}$. Denote by $M = M(s_1, s_2, \dots, s_r | t_1, t_2, \dots, t_r)$ the highway measurement (as defined in §6.1) consisting of families $\mathbb{P} = (P_1, \dots, P_r)$ of non-intersecting highway paths in $N_{n,m}$ where P_i lifts to a path on $\tilde{N}_{n,m}$ that goes from \bar{s}_i (the image of s_i in $N_{n,m}$) to \bar{t}_i . This condition fixes the homology classes $[P_i]$. Then M is a quantum cylindric loop Schur function and every quantum cylindric loop Schur function occurs in this way. For each $i \in \{1, 2, \dots, r\}$, define $a_i, b_i \in \mathbb{Z}$ by $s(P) = [a_i, b_i]$ where P is a highway path in $\tilde{N}_{n,m}$ going from s_i to t_i .

In the following, we obtain a canonical lifting to $\tilde{N}_{n,m}$ of every path P in $N_{n,m}$ considered by requiring that they start at one of the s_i .

Let $\mathbb{P} = (P_1, P_2, \dots, P_r)$ be an ordered family of highway paths on $N_{n,m}$ where P_i goes from \bar{s}_i to some \bar{t}_j , such that each of $\bar{t}_1, \bar{t}_2, \dots, \bar{t}_r$ is used once. Suppose \mathbb{P} has intersections. Among all pairs of paths (P_i, P_j) that intersect, choose the pair where $j = i + 1$ and i is minimal. Let e be the right most edge on $P_i \cap P_j$. Let $P_i = P_i^{(1)} P_i^{(2)}$ and $P_{i+1} = P_{i+1}^{(1)} P_{i+1}^{(2)}$ be the decompositions of the two paths, where $P_*^{(1)}$ contains e and all edges in P_* preceding it, while $P_*^{(2)}$ contains all edges after e . Thus $P_i^{(2)}$ and $P_{i+1}^{(2)}$ do not intersect on $N_{n,m}$. Let us suppose that $s(P_i^{(1)}) = [a, x - 1]$, $s(P_i^{(2)}) = [x, b]$, $s(P_{i+1}^{(1)}) = [c, x' - 1]$, and $s(P_{i+1}^{(2)}) = [x', d]$, where $x \equiv x' \pmod{n}$.

Let \mathbb{P}' be obtained from \mathbb{P} by setting $P'_i = P_i^{(1)}P_{i+1}^{(2)}$ and $P'_{i+1} = P_{i+1}^{(1)}P_i^{(2)}$, and leaving the rest of the paths unchanged. We compute α such that $\text{wt}(\mathbb{P}) = \epsilon^\alpha \text{wt}(\mathbb{P}')$. By Lemma 6.13 and 6.15, we have

$$\begin{aligned} \alpha &= \alpha(P_{i+1}^{(1)}, P_{i+1}^{(2)}) + \alpha(P_i^{(2)}, P_{i+1}) \\ &= (1 - \chi(c - x') - \chi(x' - 1 - d) + \chi(c - d - 1)) \\ &\quad + (-\chi(c - b - 1) + \chi(c - x) + \chi(d - x + 1)) \\ &= 1 + \chi(c - d - 1) - \chi(c - b - 1). \end{aligned} \tag{6.6}$$

Define

$$\beta(\mathbf{a}, \mathbf{b}) = \beta((a_1, b_1), (a_2, b_2), \dots, (a_r, b_r)) := \sum_{i < j | b_i < b_j} (-1 - \chi(a_j - b_j - 1) + \chi(a_j - b_i - 1))$$

and $\beta(\mathbb{P}) := \beta(s(P_1), \dots, s(P_r))$. Then (6.6) gives

$$\epsilon^{\beta(\mathbb{P})} \text{wt}(\mathbb{P}) = \epsilon^{\beta(\mathbb{P}')} \text{wt}(\mathbb{P}'). \tag{6.7}$$

Let $e(a, b) = \sum_P P$ be the weight generating function of highway paths such that $s(P) = [a, b]$. Then $\mathbb{P} \mapsto \mathbb{P}'$ sets up an involution on intersecting families of highway paths, and (6.7) allows us to cancel weights, giving

$$M = \sum_{\mathbf{b}'} (-1)^{\text{sign}(\mathbf{b}')} \epsilon^{\beta(\mathbf{a}, \mathbf{b}') - \beta(\mathbf{a}, \mathbf{b})} e(a_1, b'_1) e(a_2, b'_2) \cdots e(a_k, b'_k) \tag{6.8}$$

where the summation is over all \mathbf{b}' such that a permutation of $\sigma(\mathbf{b}')$ is congruent to \mathbf{b} modulo n and such that $\sum_i b_i = \sum_i b'_i$. Here $\text{sign}(\mathbf{b}') = \text{sign}(\sigma)$. The theorem follows from Corollary 6.3 and the observation that $e(a, b)$ is nothing but a quantum loop elementary symmetric function $e_{b-a+1}^{(a)}$. \square

Example 6.16. Let us consider the case of Example 6.12. Set $\mathbf{a} = (4, 3)$ and $\mathbf{b} = (5, 3)$. In (6.8) we have three choices of \mathbf{b}' as $(5, 3)$, $(3, 5)$ and $(6, 2)$, and obtain

$$M(\mathbf{a}, \mathbf{b}) = e(4, 5)e(3, 3) - \epsilon^{-2}e(4, 3)e(3, 5) - e(4, 6)e(3, 2).$$

By using the formulae

$$\begin{aligned} e(4, 5) &= e_2^{(4)} = q_1^{(1)}q_2^{(1)} + q_2^{(3)}q_3^{(3)} + q_1^{(1)}q_3^{(3)}, \\ e(3, 3) &= e_1^{(3)} = q_1^{(3)} + q_2^{(2)} + q_3^{(1)}, \\ e(a, a+2) &= e_3^{(a)} = q_1^{(i)}q_2^{(i)}q_3^{(i)}, \text{ for } i \equiv a \pmod{3}, \end{aligned}$$

and $e(a, a-1) = e_0^{(a)} = 1$, we see that $M(\mathbf{a}, \mathbf{b}) = s_D^{(1)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$.

7. QUANTUM CLUSTER R -MATRIX

As mentioned in the introduction, we will use Fock and Goncharov's quantization of cluster y -seeds, and we begin by writing the cluster R -matrix in y -variables.

7.1. Definition of the cluster R -matrix in y -variables. Let $\mathbb{P}_{\text{univ}}(\mathbf{y})$ denote the universal semifield generated by y_v for $v \in Q_{n,m}$. As in §3.2, for three adjacent closed vertical cycles (M^-, M, M^+) in $Q_{n,m}$, we denote the y -variables y_v for $v \in M^-, M, M^+$ by y_i^-, y_i, y_i^+ respectively, where i takes values modulo n and the arrows around the vertex $i \in M$ are given by (3.2).

Theorem 7.1. *Let $(M^-, M, M^+) = (M_{i-1}, M_i, M_{i+1})$ denote three adjacent cycles in $Q_{n,m}$. For any $j \in \mathbb{Z}/n\mathbb{Z}$, the operator $\mathcal{R}_{M,j}$ of (3.3) acts on the y -seed $(Q_{n,m}, \mathbf{y})$ by $\mathcal{R}_{M,j}(Q_{n,m}, \mathbf{y}) = (Q_{n,m}, \mathbf{y}')$, where*

$$y'_i = (\alpha_{i+2})^{-1} \cdot y_{i+1}^{-1} \cdot \alpha_i, \quad (7.1)$$

$$y'_{i-} = (\alpha_i)^{-1} \cdot y_i y_i^- \cdot \alpha_{i+1}, \quad (7.2)$$

$$y'_{i+} = (\alpha_{i+1})^{-1} \cdot y_{i+1} y_i^+ \cdot \alpha_{i+2}, \quad (7.3)$$

and the other y_v ($v \notin M \cup M^+ \cup M^-$) are unchanged. Here we set

$$\alpha_i = \alpha_i(M) := 1 + \sum_{k=1}^{n-1} y_i y_{i+1} \cdots y_{i+k-1} \in \mathbb{P}_{\text{univ}}(\mathbf{y}). \quad (7.4)$$

The action of $\mathcal{R}_{M_0,j}$ (resp. $\mathcal{R}_{M_m,j}$) are given by (7.1) and (7.3) with $M = M_0$ and $M^+ = M_1$ (resp. (7.1) and (7.2) with $M = M_m$ and $M^- = M_{m-1}$).

Theorem 7.1 follows from its quantum version Theorem 7.3, stated in the next subsection.

We define the cluster R -matrix on y -variables as the rational morphism $R_{M_j} : \mathbb{P}_{\text{univ}}(\mathbf{y}) \rightarrow \mathbb{P}_{\text{univ}}(\mathbf{y})$ for $j = 0, \dots, m$ given by Theorem 7.1.

Example 7.2. When $n = 3$, the formulae for $R_M(y_1)$, $R_M(y_1^-)$ and $R_M(y_1^+)$ are

$$\begin{aligned} R_M(y_1) &= (1 + y_3 + y_3 y_1)^{-1} y_2^{-1} (1 + y_1 + y_1 y_2), \\ R_M(y_1^-) &= (1 + y_1 + y_1 y_2)^{-1} y_1 y_1^- (1 + y_2 + y_2 y_3), \\ R_M(y_1^+) &= (1 + y_2 + y_2 y_3)^{-1} y_2 y_1^+ (1 + y_3 + y_3 y_1). \end{aligned}$$

7.2. Quantization. In the following we study the quantization of R_M using Fock-Goncharov quantization (see §2.3) corresponding to the quiver $Q_{n,m}$. Let $\mathcal{Y}_\epsilon := \mathcal{Y}_\epsilon(Q_{n,m})$ be the ambient skew field of the quantum torus (see §5.1) generated by y_{ji} ($0 \leq j \leq m$, $i \in \mathbb{Z}/n\mathbb{Z}$) with the relations:

$$y_{ji} y_{j+1,i} = \epsilon^2 y_{j+1,i} y_{ji} \quad 0 \leq j \leq m-1, \quad (7.5)$$

$$y_{ji} y_{j,i+1} = \epsilon^{-2} y_{j,i+1} y_{ji} \quad 0 \leq j \leq m, \quad (7.6)$$

$$y_{ji} y_{j+1,i-1} = \epsilon^{-2} y_{j+1,i-1} y_{ji} \quad 0 \leq j \leq m-1. \quad (7.7)$$

For any closed vertical cycle M in $Q_{n,m}$ and $j \in \mathbb{Z}/n\mathbb{Z}$, we define a sequence of a permutation and $2n - 2$ quantum mutations by

$$\mathcal{R}_{M,j}^\epsilon := s_{j-2,j-1} \circ (\mu_j^\epsilon \circ \mu_{j+1}^\epsilon \cdots \circ \mu_{j-4}^\epsilon \circ \mu_{j-3}^\epsilon) \circ (\mu_{j-1}^\epsilon \circ \mu_{j-2}^\epsilon \circ \cdots \circ \mu_{j+1}^\epsilon \circ \mu_j^\epsilon).$$

Theorem 7.3. Let $(M^-, M, M^+) = (M_{i-1}, M_i, M_{i+1})$ denote three adjacent cycles in $Q_{n,m}$. Then, for any j we have $\mathcal{R}_{M,j}^\epsilon(Q_{n,m}, \mathbf{y}) = (Q_{n,m}, \mathbf{y}')$ where

$$y'_i = (\alpha_{i+2}^\epsilon)^{-1} \cdot y_{i+1}^{-1} \cdot \alpha_i^\epsilon, \quad (7.8)$$

$$y'_{i-} = (\alpha_i^\epsilon)^{-1} \cdot \epsilon y_i y_i^- \cdot \alpha_{i+1}^\epsilon, \quad (7.9)$$

$$y'_{i+} = (\alpha_{i+1}^\epsilon)^{-1} \cdot \epsilon y_{i+1} y_i^+ \cdot \alpha_{i+2}^\epsilon, \quad (7.10)$$

and the other y_v ($v \notin \cup M^- \cup M \cup M^+$) are unchanged. Here we set

$$\alpha_i^\epsilon = \alpha_i^\epsilon(M) := 1 + \sum_{k=1}^{n-1} \epsilon^k y_i y_{i+1} \cdots y_{i+k-1} \in \mathcal{Y}_\epsilon.$$

The action of $\mathcal{R}_{M_0,j}^\epsilon$ (resp. $\mathcal{R}_{M_m,j}^\epsilon$) is given by (7.8) and (7.10) with $M = M_0$ and $M^+ = M_1$ (resp. (7.8) and (7.9) with $M = M_m$ and $M^- = M_{m-1}$).

Example 7.4. We demonstrate the calculation of $\mathcal{R}_{M,1}^\epsilon(Q_{n,m}, \mathbf{y}) = (Q_{n,m}, \mathbf{y}')$ in the case of $n = 3$. Due to Proposition 8.8 we have a sequence of seeds as

$$(Q_{3,m}, \mathbf{y}) \xrightarrow{\mu_1^\epsilon} (Q_1, \mathbf{y}[1]) \xrightarrow{\mu_2^\epsilon} (Q_2, \mathbf{y}[2]) \xrightarrow{\mu_3^\epsilon} (Q_3, \mathbf{y}[3]) \xrightarrow{s_{2,3}} (Q_1, \mathbf{y}[\bar{1}]) \xrightarrow{\mu_1^\epsilon} (Q_{3,m}, \mathbf{y}').$$

In the following we write only y -variables y_v of $v \in M^- \cup M \cup M^+$. By starting with

$$\mathbf{y} = \begin{pmatrix} y_{1-} & \textcolor{red}{y}_1 & y_{1+} \\ y_{2-} & y_2 & y_{2+} \\ y_{3-} & y_3 & y_{3+} \end{pmatrix},$$

we obtain

$$\begin{aligned} \mathbf{y}[1] &= \begin{pmatrix} y_1^-(1 + \epsilon y_1^{-1})^{-1} & y_1^{-1} & y_1^+(1 + \epsilon y_1) \\ y_2^-(1 + \epsilon y_1) & \textcolor{red}{y}_2(1 + \epsilon y_1^{-1})^{-1} & y_2^+ \\ y_3^- & y_3(1 + \epsilon y_1) & y_3^+(1 + \epsilon y_1^{-1})^{-1} \end{pmatrix}, \\ \mathbf{y}[2] &= \begin{pmatrix} y_1^-(1 + \epsilon y_1^{-1})^{-1} & y_1^{-1}(1 + \epsilon y_2[1]^{-1})^{-1} & y_1^+(1 + \epsilon y_1) \\ y_2^-(1 + \epsilon y_1) & y_2[1]^{-1} & y_2^+(1 + \epsilon y_2[1]) \\ y_3^-(1 + \epsilon y_2[1]) & \textcolor{red}{y}_3(1 + \epsilon y_1) & y_3^+(1 + \epsilon y_1^{-1})^{-1} \end{pmatrix}, \\ \mathbf{y}[3] &= \begin{pmatrix} y_1^-(1 + \epsilon y_1^{-1})^{-1} & y_1^{-1}(1 + \epsilon y_2[1]^{-1})^{-1}(1 + \epsilon y_3[2]) & y_1^+(1 + \epsilon y_1) \\ y_2^-(1 + \epsilon y_1) & y_2[1]^{-1} & y_2^+(1 + \epsilon y_2[1])(1 + \epsilon y_3[2]^{-1})^{-1} \\ y_3^-(1 + \epsilon y_2[1])(1 + \epsilon y_3[2]^{-1})^{-1} & y_3[2]^{-1} & y_3^+(1 + \epsilon y_1^{-1})^{-1} \end{pmatrix}, \\ \mathbf{y}[\bar{1}] &= \begin{pmatrix} y_1^-(1 + \epsilon y_1^{-1})^{-1} & \textcolor{red}{y}_1^{-1}(1 + \epsilon y_2[1]^{-1})^{-1}(1 + \epsilon y_3[2]) & y_1^+(1 + \epsilon y_1) \\ y_2^-(1 + \epsilon y_1) & y_3[2]^{-1} & y_2^+(1 + \epsilon y_2[1])(1 + \epsilon y_3[2]^{-1})^{-1} \\ y_3^-(1 + \epsilon y_2[1])(1 + \epsilon y_3[2]^{-1})^{-1} & y_2[1]^{-1} & y_3^+(1 + \epsilon y_1^{-1})^{-1} \end{pmatrix}, \\ \mathbf{y}' &= \begin{pmatrix} y_1^-(1 + \epsilon y_1^{-1})^{-1}(1 + \epsilon y_1[3]) & y_1[3]^{-1} & y_1^+(1 + \epsilon y_1)(1 + \epsilon y_1[3]^{-1})^{-1} \\ y_2^-(1 + \epsilon y_1)(1 + \epsilon y_1[3]^{-1})^{-1} & y_3[2]^{-1}(1 + \epsilon y_1[3]) & y_2^+(1 + \epsilon y_2[1])(1 + \epsilon y_3[2]^{-1})^{-1} \\ y_3^-(1 + \epsilon y_2[1])(1 + \epsilon y_3[2]^{-1})^{-1} & y_2[1]^{-1}(1 + \epsilon y_1[3]^{-1})^{-1} & y_3^+(1 + \epsilon y_1^{-1})^{-1}(1 + \epsilon y_1[3]) \end{pmatrix}, \end{aligned}$$

Here the y -variables corresponding to mutated vertices are indicated in red. By arranging \mathbf{y}' , we obtain

$$\begin{aligned} R_M^\epsilon(y_1) &= (1 + \epsilon y_3 + \epsilon^2 y_3 y_1)^{-1} y_2^{-1} (1 + \epsilon y_1 + \epsilon^2 y_1 y_2), \\ R_M^\epsilon(y_1^-) &= (1 + \epsilon y_1 + \epsilon^2 y_1 y_2)^{-1} \epsilon y_1 y_1^- (1 + \epsilon y_2 + \epsilon^2 y_2 y_3), \\ R_M^\epsilon(y_1^+) &= (1 + \epsilon y_2 + \epsilon^2 y_2 y_3)^{-1} \epsilon y_2 y_1^+ (1 + \epsilon y_3 + \epsilon^2 y_3 y_1). \end{aligned}$$

In the limit of $\epsilon \rightarrow 1$, these reduce to Example 7.2.

The following is the key lemma in the proof of Theorem 7.3.

Lemma 7.5. *Assume that the y -variables lie in the tropical semifield $\mathbb{P}_{\text{trop}}(\mathbf{y})$. Then for any $j \in \mathbb{Z}/n\mathbb{Z}$, the operator $\mathcal{R}_{M,j}$ defined at (3.3) acts on the initial y -seed $(Q_{n,m}, \mathbf{y})$ as $\mathcal{R}_{M,j}(Q_{n,m}, \mathbf{y}) = (Q_{n,m}, \mathbf{y}')$, where*

$$y'_v = \begin{cases} (y_{i+1})^{-1} & \text{if } v = i \in M, \\ y_i y_i^- & \text{if } v = i^- \in M^-, \\ y_{i+1} y_i^+ & \text{if } v = i^+ \in M^+, \\ y_v & \text{otherwise.} \end{cases} \quad (7.11)$$

Proof. Due to Proposition 8.8, we compute $\mathcal{R}_{M,1}(Q_{n,m}, \mathbf{y})$ by the following sequence of y -seeds

$$\begin{aligned} (Q_0, \mathbf{y}[0]) &= (Q_{n,m}, \mathbf{y}) \xrightarrow{\mu_1} (Q_1, \mathbf{y}[1]) \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_n} (Q_n, \mathbf{y}[n]) \xrightarrow{s_{n-1,n}} (Q_{n-2}, \mathbf{y}[\overline{n-2}]) \\ &\xrightarrow{\mu_{n-2}} (Q_{n-3}, \mathbf{y}[\overline{n-3}]) \xrightarrow{\mu_{n-3}} \cdots \xrightarrow{\mu_1} (Q_0, \mathbf{y}[\overline{0}]) = \mathcal{R}_{M,1}(Q_{n,m}, \mathbf{y}). \end{aligned} \quad (7.12)$$

Using the lemmas in §8.1, we explicitly calculate the change of tropical y -variables in (7.12) by using the following general fact: assume that a quiver Q' has only simply weighted arrows, and consider a tropical mutation $(Q'', \mathbf{y}'') = \mu_j(Q', \mathbf{y}')$ at $j \in Q'$. If the y -variable at the mutation point, y'_j , is a nonnegative power of the initial y -variables, then $\mathbf{y}'' = (y''_i)_i$ is determined by the following rule:

$$y''_i = \begin{cases} y'_i y'_j & \text{if } j \rightarrow i \text{ in } Q', \\ y'_i & \text{otherwise.} \end{cases}$$

In the following we write only $y_v[j]$ of $v \in M^- \cup M \cup M^+$ as

$$\mathbf{y}[0] = \begin{pmatrix} y_{1-} & \textcolor{red}{y_1} & y_{1+} \\ \vdots & \vdots & \vdots \\ y_{n-} & y_n & y_{n+} \end{pmatrix},$$

where the mutation point in the sequence (7.12) is indicated in red. Let $y_{[i,j]} := y_i y_{i+1} \cdots y_j$. By direct calculation we obtain the following, where one sees that the

‘if’ part of the above fact always holds in the sequence of mutations (7.12).

$$\begin{aligned}
\mathbf{y}[k] &= \begin{pmatrix} y_{1-}y_1 & y_2 & y_{1+} \\ y_{2-} & y_3 & y_{2+} \\ \vdots & \vdots & \vdots \\ y_{k-1-} & y_k & y_{k-1+} \\ y_k^- & y_{[1,k]}^{-1} & y_k^+ \\ y_{k+1-} & \textcolor{red}{y}_{[1,k+1]} & y_{k+1+} \\ y_{k+2-} & y_{k+2} & y_{k+2+} \\ \vdots & \vdots & \vdots \\ y_{n-1-} & y_{n-1} & y_{n-1+} \\ y_n^- & y_n & y_{n+}y_1 \end{pmatrix} \quad \text{for } k = 1, 2, \dots, n-2, \\
\mathbf{y}[n-1] &= \begin{pmatrix} y_{1-}y_1 & y_2 & y_{1+} \\ y_{2-} & y_3 & y_{2+} \\ \vdots & \vdots & \vdots \\ y_{n-2-} & y_{n-1} & y_{n-2+} \\ y_{n-1-} & y_{[1,n-1]}^{-1} & y_{n-1+} \\ y_n^- & \textcolor{red}{y}_n & y_{n+}y_1 \end{pmatrix}, \quad \mathbf{y}[n] = \begin{pmatrix} y_{1-}y_1 & y_2 & y_{1+} \\ y_{2-} & y_3 & y_{2+} \\ \vdots & \vdots & \vdots \\ y_{n-2-} & y_{n-1} & y_{n-2+} \\ y_{n-1-} & y_{[1,n-1]}^{-1} & y_{n-1+}y_n \\ y_n^- y_n & y_n^{-1} & y_{n+}y_1 \end{pmatrix}, \\
\mathbf{y}[\bar{k}] &= \begin{pmatrix} y_{1-}y_1 & y_2 & y_{1+} \\ y_{2-} & y_3 & y_{2+} \\ \vdots & \vdots & \vdots \\ y_k^- & \textcolor{red}{y}_{k+1} & y_k^+ \\ y_{k+1-} & y_{k+2}^{-1} & y_{k+1+}y_{k+2} \\ y_{k+2-}y_{k+2} & y_{k+3}^{-1} & y_{k+2+}y_{k+3} \\ \vdots & \vdots & \vdots \\ y_{n-1-}y_{n-1} & y_n^{-1} & y_{n-1+}y_n \\ y_n^- y_n & y_{[1,k+1]}^{-1} & y_{n+}y_1 \end{pmatrix} \quad \text{for } k = n-2, n-3, \dots, 0.
\end{aligned}$$

From $\mathbf{y}[\bar{0}]$ we obtain (7.11) for the $j = 1$ case, which is obviously independent of j . \square

Proof of Theorem 7.3. First we suppose that M is not equal to either M_0 or M_m . We use the notations $(Q_k, \mathbf{y}[k])$ and $(Q_k, \mathbf{y}[\bar{k}])$ defined in the proof of Lemma 7.5. From Theorem 2.1 and Lemma 7.5, it follows that $\mathcal{R}_{M,j}^\epsilon(Q_{n,m}, \mathbf{y})$ is independent of j . Therefore it is enough to check (7.8)–(7.10) for one i of each. We explicitly calculate $y_{n-}[\bar{0}]$, $y_{n-1+}[\bar{0}]$ and $y_{n-1}[\bar{0}]$.

From the lemmas in §8.1, we obtain

$$y_{n-}[\bar{0}] = y_{n-}[n] = y_{n-}(1 + \epsilon y_{n-1}[n-2])(1 + \epsilon y_n[n-1]^{-1})^{-1}, \quad (7.13)$$

$$y_{n-1+}[\bar{0}] = y_{n-1+}[n] = y_{n-1+}(1 + \epsilon y_{n-1}[n-2])(1 + \epsilon y_n[n-1]^{-1})^{-1}, \quad (7.14)$$

$$y_{n-1}[\bar{0}] = y_{n-1}[\overline{n-3}] = y_n[n-1]^{-1} (1 + \epsilon y_{n-2}[\overline{n-2}]), \quad (7.15)$$

and

$$y_{j+1}[j] = y_{j+1} (1 + \epsilon y_j[j-1]^{-1})^{-1} \quad j = 1, \dots, n-2, \quad (7.16)$$

$$y_n[j] = y_n[j-1](1 + \epsilon y_j[j-1]) \quad j = 1, \dots, n-2, \quad (7.17)$$

$$y_{n-2}[\overline{n-2}] = y_{n-2}[n] = y_{n-2}[n-3]^{-1}(1 + \epsilon y_{n-1}[n-2]^{-1})^{-1}(1 + \epsilon y_n[n-1]). \quad (7.18)$$

By using (7.16) recursively, we obtain

$$(1 + \epsilon y_1)(1 + \epsilon y_2[1]) \cdots (1 + \epsilon y_j[j-1]) = 1 + \sum_{k=1}^j \epsilon^k y_1 y_2 \cdots y_k,$$

for $j = 1, \dots, n-1$, and it follows that

$$y_{n-1}[n-2] = \epsilon^{n-2}(\alpha'_1)^{-1} y_1 y_2 \cdots y_{n-1}, \quad (7.19)$$

$$y_{n-2}[n-3] = \epsilon^{n-3}(\alpha''_1)^{-1} y_1 \cdots y_{n-2}. \quad (7.20)$$

Here we set $\alpha'_1 = \alpha_1 \epsilon - \epsilon^{n-1} y_1 \cdots y_{n-1}$, $\alpha''_1 = \alpha'_1 - \epsilon^{n-2} y_1 \cdots y_{n-2}$. Furthermore, from (7.17) and the fact that $y_n[n-1] = y_n[n-2]$ we obtain

$$y_n[n-1] = y_n \alpha'_1. \quad (7.21)$$

Let us calculate $y_{n-}[\bar{0}]$, $y_{n-1+}[\bar{0}]$ and $y_{n-1}[\bar{0}]$. The quiver Q_{n-2} tells us (see §2.3) that $y_n[n-2]$ and $y_{n-1}[n-2]$ commute and that $y_n[n-2] \cdot y_{n-} = \epsilon^{-2} y_{n-} \cdot y_n[n-2]$. Thus, (7.13) with (7.19) and (7.21) gives

$$\begin{aligned} y_{n-}[\bar{0}] &= (1 + \epsilon^{-1} y_n[n-1]^{-1})^{-1} y_{n-} (1 + \epsilon y_{n-1}[n-2]) \\ &= (1 + \epsilon^{-1} (y_n \alpha'_1)^{-1})^{-1} y_{n-} (1 + \epsilon^{n-1} (\alpha'_1)^{-1} y_1 y_2 \cdots y_{n-1}) \\ &= (\alpha_n^\epsilon)^{-1} \cdot \epsilon y_n y_{n-} \cdot \alpha_1^\epsilon, \end{aligned}$$

where we have used that $\alpha_n^\epsilon = 1 + \epsilon y_n \alpha'_1$ and that α'_1 commutes with y_{n-} . In a similar way, from (7.14) we obtain $y_{n-1+}[\bar{0}] = (\alpha_n^\epsilon)^{-1} \cdot \epsilon y_n y_{n-1+} \cdot \alpha_1^\epsilon$. From (7.15) and (7.18), we get

$$y_{n-1}[\bar{0}] = [1 + \epsilon^{-1} (1 + \epsilon^{-1} y_{n-1}[n-2]^{-1})^{-1} y_{n-2}[n-3]^{-1} (1 + \epsilon y_n[n-2])] y_n[n-2]^{-1}.$$

By substituting (7.19)–(7.21), $y_{n-1}[\bar{0}]$ is calculated to be

$$\begin{aligned} & (y_n \alpha'_1)^{-1} + \epsilon^{-1} \underbrace{(1 + \epsilon^{-n+1} (y_1 \cdots y_{n-1})^{-1} \alpha'_1)^{-1} \epsilon^{-n+3} (y_1 \cdots y_{n-2})^{-1} \alpha''_1 ((y_n \alpha'_1)^{-1} + \epsilon)}_{\text{key part}} \\ &= (y_n \alpha'_1)^{-1} + \epsilon^{-1} (\alpha_1^\epsilon)^{-1} \epsilon^{n-1} \underbrace{(y_1 \cdots y_{n-1}) \epsilon^{-n+3} (y_1 \cdots y_{n-2})^{-1} \alpha''_1 ((y_n \alpha'_1)^{-1} + \epsilon)}_{\text{key part}} \\ &= (\alpha_1^\epsilon)^{-1} [(\alpha'_1 + \epsilon^{n-1} y_1 \cdots y_{n-1}) (y_n \alpha'_1)^{-1} + \epsilon^{-1} y_{n-1} \alpha''_1 ((y_n \alpha'_1)^{-1} + \epsilon)] \\ &= (\alpha_1^\epsilon)^{-1} [y_n^{-1} + \epsilon^{-1} y_{n-1} \alpha'_1 (y_n \alpha'_1)^{-1} + y_{n-1} \alpha''_1] \\ &= (\alpha_1^\epsilon)^{-1} [y_n^{-1} + q^{-1} y_{n-1} y_n^{-1} + y_{n-1} \alpha''_1] = (\alpha_1^\epsilon)^{-1} y_n^{-1} \alpha_{n-1}^\epsilon. \end{aligned}$$

Here the underlined parts denote key parts of the calculation. This completes the proof for $M = M_k$ where $k = 1, 2, \dots, m-1$.

Finally, the last statement of the theorem follows from the following general fact: suppose Q is a quiver and $\tilde{Q} \subset Q$ is the quiver restricted to some subset of vertices. Let $A = \mu_{i_1} \circ \cdots \circ \mu_{i_r}$ be a sequence of mutations where $i_1, i_2, \dots, i_r \in \tilde{Q}$. Then $A(Q)$ restricted to the vertices in \tilde{Q} is the same quiver as $A(\tilde{Q})$. We apply this to $M \cup M^+ \subset M^- \cup M \cup M^+$ (resp. $M^- \cup M \subset M^- \cup M \cup M^+$) to obtain the statement for $M = M_0$ (resp. $M = M_m$). \square

7.3. Yang-Baxter relation. For $j = 1, \dots, m-1$ we define the quantum cluster R -matrix on y -variables as the morphism $R_{M_j}^\epsilon: \mathcal{Y}_\epsilon \rightarrow \mathcal{Y}_\epsilon$ given by Theorem 7.3.

Theorem 7.6. *We have $R_{M_j}^\epsilon R_{M_{j+1}}^\epsilon R_{M_j}^\epsilon = R_{M_{j+1}}^\epsilon R_{M_j}^\epsilon R_{M_{j+1}}^\epsilon$ and $(R_j^\epsilon)^2 = \text{id}$. Furthermore, for i, j such that $|i - j| > 1$ we have $R_{M_i}^\epsilon R_{M_j}^\epsilon = R_{M_j}^\epsilon R_{M_i}^\epsilon$. Thus $R_{M_j}^\epsilon$ generate an action of \mathfrak{S}_m on \mathcal{Y}_ϵ .*

Remark 7.7. If we include $R_{M_0}^\epsilon$ and $R_{M_m}^\epsilon$, Theorem 7.6 extends to give an action of \mathfrak{S}_{m+2} .

Proof. The equality $(R_j^\epsilon)^2 = \text{id}$ follows from the involutivity of quantum mutations.

Suppose $j \in \{1, \dots, m-2\}$. Set $M = M_j$ and consider four adjacent cycles M^- , M , M^+ and M^{++} in this order on $Q_{n,m}$. Due to Theorem 2.1, to prove $R_M R_{M^+} R_M = R_{M^+} R_M R_{M^+}$ it is enough to show that the relation $R_M^\epsilon R_{M^+}^\epsilon R_M^\epsilon = R_{M^+}^\epsilon R_M^\epsilon R_{M^+}^\epsilon$ hold for the initial tropical y -seed $(Q_{n,m}, \mathbf{y})$.

In the following, we explicitly write only y_v of $v \in M^- \cup M \cup M^+ \cup M^{++}$, since the quiver $Q_{n,m}$ and the others y -variables are unchanged by R_M and R_{M^+} . By using (7.11) we calculate $R_M R_{M^+} R_M(\mathbf{y})$:

$$\begin{aligned} (y_{i-}, y_i, y_{i+}, y_{i++}) &\xrightarrow{R_M} (y_{i-} y_i, y_{i+1}^{-1}, y_{i+1} y_{i+}, y_{i++}) \\ &\xrightarrow{R_{M^+}} (y_i y_{i-}, y_{i+}, (y_{i+2} y_{i+1})^{-1}, y_{i+2} y_{i+1} y_{i++}) \\ &\xrightarrow{R_M} (y_i y_{i-} y_{i+}, y_{i+1}^{-1}, y_{i+2}^{-1}, y_{i+2} y_{i+1} y_{i++}). \end{aligned}$$

Next, we calculate $R_{M^+} R_M R_{M^+}(\mathbf{y})$:

$$\begin{aligned} (y_{i-}, y_i, y_{i+}, y_{i++}) &\xrightarrow{R_{M^+}} (y_{i-}, y_i y_{i+}, y_{i+1}^{-1}, y_{i+1} y_{i++}) \\ &\xrightarrow{R_M} (y_{i-} y_i y_{i+}, (y_{i+1} y_{i+1})^{-1}, y_{i+1}, y_{i+1} y_{i++}) \\ &\xrightarrow{R_{M^+}} (y_{i-} y_i y_{i+}, y_{i+1}^{-1}, y_{i+2}^{-1}, y_{i+2} y_{i+1} y_{i++}). \end{aligned}$$

Thus $R_M R_{M^+} R_M(\mathbf{y}) = R_{M^+} R_M R_{M^+}(\mathbf{y})$ holds.

When M and M' are not adjacent, the identity $R_M^\epsilon R_{M'}^\epsilon = R_{M'}^\epsilon R_M^\epsilon$ is immediate. \square

7.4. From quantum geometric R to quantum cluster R . In this section, we show that the quantum geometric R -matrix R_j^ϵ and the quantum cluster R -matrix $R_{M_j}^\epsilon$ are compatible. In particular, we shall show that Theorem 5.3 follows from Theorem 7.6.

Let $\mathcal{Y}'_\epsilon \subset \mathcal{Y}_\epsilon$ be the skew subfield generated by y_{ji} where $j \in 1, 2, \dots, m-1$. We note that \mathcal{Y}'_ϵ is closed under the operations $R_{M_1}^\epsilon, R_{M_2}^\epsilon, \dots, R_{M_{m-1}}^\epsilon$.

Proposition 7.8. *The following transformation ϕ induces an embedding of $\mathbb{C}[\epsilon]$ -algebras $\phi_\epsilon: \mathcal{Y}'_\epsilon \rightarrow \mathcal{Q}_\epsilon(N_{n,m})$:*

$$y_{ji} \mapsto \epsilon^{-1} q_{j,i}^{-1} q_{j+1,i-1} \quad j = 1, \dots, m-1, \quad i \in \mathbb{Z}/n\mathbb{Z} \quad (7.22)$$

This proposition is easily proved by direct calculation. We omit the proof.

Theorem 7.9. *For $j = 1, \dots, m-1$, we have $R_j^\epsilon \circ \phi_\epsilon = \phi_\epsilon \circ R_{M_j}^\epsilon$. Thus the quantum geometric R -matrix and the quantum cluster R -matrix are compatible.*

Proof. Let $(W^{--}, W^-, W, W^+) = (W_{j-1}, W_j, W_{j+1}, W_{j+2})$ be four adjacent vertical cycles on $N_{n,m}$ where q_i^{--} , q_i^- , q_i and q_i^+ are located respectively, and let $(M^-, M, M^+) = (M_{j-1}, M_j, M_{j+1})$ be the corresponding three adjacent closed cycles on $Q_{n,m}$ where y_i^- ,

y_i and y_i^+ are located respectively. It is enough to show that $R_j^\epsilon \circ \phi_\epsilon(y_v) = \phi_\epsilon \circ R_{M_j}^\epsilon(y_v)$ for $v \in M^- \cup M \cup M^+$.

The map ϕ_ϵ sends the y -variables on (M^-, M, M^+) to

$$y_i^- \mapsto \epsilon^{-1}(q_i^-)^{-1} q_{i-1}^-, \quad y_i \mapsto \epsilon^{-1}(q_i^-)^{-1} q_{i-1}, \quad y_i^+ \mapsto \epsilon^{-1}(q_i)^{-1} q_{i-1}^+.$$

We use the relation

$$\phi_\epsilon(\alpha_{i+1}^\epsilon(M)) = (q_{i-1}^- q_{i-2}^- \cdots q_{i-n+1}^-)^{-1} \kappa_i^\epsilon(\mathbf{q}^-, \mathbf{q}),$$

which can be easily checked.

We compute

$$\begin{aligned} \phi_\epsilon(R_M^\epsilon(y_i)) &= \phi_\epsilon((\alpha_{i+2}^\epsilon)^{-1} y_{i+1}^{-1} \alpha_i^\epsilon) \\ &= (\kappa_{i+1}^\epsilon)^{-1} q_i^- q_{i-1}^- \cdots q_{i-n+2}^- \cdot \epsilon q_i^{-1} q_{i+1}^- \cdot (q_{i-2}^- q_{i-3}^- \cdots q_{i-n}^-)^{-1} \kappa_{i-1}^\epsilon \\ &= (\kappa_{i+1}^\epsilon)^{-1} \epsilon^{-1} q_i^{-1} q_{i-1}^- \kappa_{i-1}^\epsilon \\ &= \epsilon^{-1} \cdot ((\kappa_i^\epsilon)^{-1} q_i \kappa_{i+1}^\epsilon)^{-1} \cdot ((\kappa_i^\epsilon)^{-1} q_{i-1}^- \kappa_{i-1}^\epsilon) \\ &= \epsilon^{-1} R_j^\epsilon(q_i^-)^{-1} \cdot R_j^\epsilon(q_{i-1}) \\ &= R_j^\epsilon(\phi_\epsilon(y_i)) \end{aligned}$$

and

$$\begin{aligned} \phi_\epsilon(R_M^\epsilon(y_i^+)) &= \phi_\epsilon((\alpha_{i+1}^\epsilon)^{-1} \cdot \epsilon y_{i+1} y_i^+ \cdot \alpha_{i+2}^\epsilon) \\ &= (\kappa_i^\epsilon)^{-1} q_{i-1}^- q_{i-2}^- \cdots q_{i-n+1}^- \cdot \epsilon^{-1} (q_{i+1}^-)^{-1} q_{i-1}^+ \cdot (q_i^- q_{i-1}^- \cdots q_{i-n+2}^-)^{-1} \kappa_{i+1}^\epsilon \\ &= \epsilon^{-1} (\kappa_i^\epsilon)^{-1} (q_i^-)^{-1} \kappa_{i+1}^\epsilon q_{i-1}^+ \\ &= \epsilon^{-1} R_j^\epsilon(q_i)^{-1} \cdot q_{i-1}^+ \\ &= R_j^\epsilon(\phi_\epsilon(y_i^+)). \end{aligned}$$

The calculation for y_i^- is similar. □

The following is an immediate consequence of Theorem 7.9.

Corollary 7.10. *Let \mathcal{Y}' be the $\epsilon \rightarrow 1$ limit of \mathcal{Y}'_ϵ , i.e., the rational function field over \mathbb{C} generated by (commutative) y -variables y_{ji} ($j = 1, \dots, m-1$, $i \in \mathbb{Z}/n\mathbb{Z}$). Let ϕ be the embedding map $\phi: \mathcal{Y}' \rightarrow \mathcal{Q}(N_{n,m})$ given by $y_{ji} \mapsto q_{j,i}^{-1} q_{j+1,i-1}$. Then, the classical version of Theorem 7.9 also holds: for $j = 1, \dots, m-1$ we have $R_j \circ \phi = \phi \circ R_{M_j}$.*

By combining Theorem 4.2 and Corollary 7.10, we get to the following:

Theorem 7.11. *We have a commutative diagram:*

$$\begin{array}{ccccc} \mathcal{Y}' & \xrightarrow{\phi} & \mathcal{Q}_{N_{n,m}} & \xrightarrow{\iota_m} & \mathcal{F}(\tilde{Q}'_{n,m}) \\ R_{M_j} \downarrow & & R_j \downarrow & & \downarrow \tilde{R}_{M_j} \\ \mathcal{Y}' & \xrightarrow{\phi} & \mathcal{Q}(N_{n,m}) & \xrightarrow{\iota_m} & \mathcal{F}(\tilde{Q}'_{n,m}) \end{array}$$

for $j = 1, \dots, m-1$, where ι_m is given by (4.2). In particular, we obtain the expression of y_{ji} in $\mathcal{F}(\tilde{Q}'_{n,m})$,

$$\iota_m \circ \phi(y_{ji}) = \frac{x_{j-1,i+1} x_{j,i-1} x_{j+1,i}}{x_{j-1,i} x_{j,i+1} x_{j+1,i-1}} \cdot \frac{X_{v_{(j,i),(j+1,i-1)}}}{X_{v_{(j-1,i+1),(j,i)}}}. \quad (7.23)$$

We remark that the formula (7.23) can be written as follows: for a non-frozen vertex v in $\tilde{Q}'_{n,m}$ we have

$$\iota_m \circ \phi(y_v) = \prod_{u \in \tilde{Q}'_{n,m}} x_u^{b_{uv}}. \quad (7.24)$$

Here $B = (b_{uv})$ is the exchange matrix corresponding to $\tilde{Q}'_{n,m}$, and in the right-hand side we take both frozen and non-frozen vertices for u . See Figure 6. Equation (7.24) should, for example, be compared to [FG09a, (11)].

7.5. Proof of Theorem 5.2. Assume we are applying R_j^ϵ . We saw in the proof of Proposition 6.8 that $q_{j,i+1}$ and $q_{j+1,i}$ satisfy the same commutation relations with any $q_{k,i}$ where $k \neq j, j+1$ as $(\kappa_{i+1}^\epsilon)^{-1}$ does. Let q be such a $q_{k,i}$. Recall that for monomials $m, m' \in \mathcal{Q}_\epsilon$, we have defined the integer $\alpha(m, m')$ by $mm' = \epsilon^{\alpha(m, m')} m'm$. We may use the same notation even when m is not a monomial, as long as m, m' commute up to a power of ϵ .

We see from the definition of $R_j^\epsilon(q_{i,j+1})$ that

$$\begin{aligned} \alpha(R_j^\epsilon(q_{j+1,i}), q) &= \alpha((\kappa_{i+1}^\epsilon)^{-1}, q) + \alpha(q_{i,j}, q) + \alpha(\kappa_i^\epsilon, q) \\ &= \alpha(q_{j+1,i}, q) + \alpha(q_{j,i}, q) - \alpha(q_{j,i}, q) = \alpha(q_{j+1,i}, q). \end{aligned}$$

We obtain $\alpha(R_j^\epsilon(q_{j,i}), q) = \alpha(q_{j,i}, q)$ in the same way.

It remains to argue that the commutation relations are preserved between the parameters in \mathbf{q}_j and \mathbf{q}_{j+1} .

First we show $\alpha(q_{j,i}, q_{j+1,i})$ is preserved. Recall the notations in the proof of Proposition 6.8, and write p_i and q_i for $q_{j,i}$ and $q_{j+1,i}$ respectively. By a direct calculation we have

$$\begin{aligned} p_i \kappa_i^\epsilon &= \epsilon^2 \kappa_i^\epsilon p_i + (1 - \epsilon^2) p_{i-1} p_{i-2} \cdots p_i, \\ q_i \kappa_i^\epsilon &= \epsilon^{-1} \kappa_i^\epsilon q_i + (\epsilon - \epsilon^{-1}) p_{i-1} p_{i-2} \cdots p_{i+1} q_i. \end{aligned}$$

Therefore we see that $q_i p_i \cdot \kappa_i^\epsilon = \epsilon \kappa_i^\epsilon \cdot q_i p_i$. By using this formula and (6.4), we obtain

$$\begin{aligned} p'_i q'_i &= (\kappa_i^\epsilon)^{-1} q_i p_i \kappa_i^\epsilon = \epsilon q_i p_i, \\ q'_i p'_i &= (\kappa_{i+1}^\epsilon)^{-1} p_i q_i \kappa_{i+1}^\epsilon = \epsilon^{-1} p_i q_i. \end{aligned}$$

Hence $\alpha(p'_i, q'_i) = \alpha(p_i, q_i) = 1$ follows.

Next we show that the other cases are preserved. For elements $m, m' \in \mathcal{Y}_\epsilon$, we define $\alpha(m, m')$ by $mm' = \epsilon^{\alpha(m, m')} m'm$, whenever m, m' commute up to a power of ϵ . Let $\mathbf{y}' = R_{M_j}^\epsilon(\mathbf{y})$. We have $\alpha(y'_{h,i}, y'_{j,k}) = \alpha(y_{h,i}, y_{j,k})$ since quantum mutations induce morphisms of quantum tori.

By Theorem 7.9 and $\alpha(y'_{j,i}, y'_{j+1,i}) = \alpha(y_{j,i}, y_{j+1,i})$ we obtain

$$\begin{aligned} &\alpha(q'_{j,i}, q'_{j+1,i}) - \alpha(q'_{j,i}, q'_{j+2,i-1}) - \alpha(q'_{j+1,i-1}, q'_{j+1,i}) + \alpha(q'_{j+1,i-1}, q'_{j+2,i-1}) \\ &= \alpha(q_{j,i}, q_{j+1,i}) - \alpha(q_{j,i}, q_{j+2,i-1}) - \alpha(q_{j+1,i-1}, q_{j+1,i}) + \alpha(q_{j+1,i-1}, q_{j+2,i-1}) \end{aligned}$$

assuming that all the parameters mentioned quasicommute. But we have already shown that the first, second and fourth terms on each side of the equality are equal. Thus it follows that $q'_{j+1,i-1}$ and $q'_{j+1,i}$ quasicommute and that $\alpha(q'_{j+1,i-1}, q'_{j+1,i}) = \alpha(q_{j+1,i-1}, q_{j+1,i})$.

In the same manner, from $\alpha(y'_{j-1,i}, y'_{j,i}) = \alpha(y_{j-1,i}, y_{j,i})$ it follows that $\alpha(q'_{j,i-1}, q'_{j,i}) = \alpha(q_{j,i-1}, q_{j,i})$. This computes $\alpha(q, \tilde{q})$ whenever q and \tilde{q} have distance one on $N_{n,m}$.

Next, we consider pairs of parameters that have distance two on $N_{n,m}$. From $\alpha(y'_{j,i}, y'_{j+1,i-1}) = \alpha(y_{j,i}, y_{j+1,i-1})$ we obtain $\alpha(q'_{j,i}, q'_{j+1,i-1}) = \alpha(q_{j,i}, q_{j+1,i-1})$ and from $\alpha(y'_{j-1,i}, y'_{j+1,i}) = \alpha(y_{j-1,i}, y_{j+1,i})$ we obtain $\alpha(q'_{j,i-1}, q'_{j+1,i}) = \alpha(q_{j,i-1}, q_{j+1,i})$. Also, from $\alpha(y'_{j,i}, y'_{j-1,i-1}) = \alpha(y_{j,i}, y_{j-1,i-1})$ we obtain $\alpha(q'_{j,i}, q'_{j,i-2}) = \alpha(q_{j,i}, q_{j,i-2})$ and from $\alpha(y'_{j+1,i}, y'_{j,i-1}) = \alpha(y_{j+1,i}, y_{j,i-1})$ we obtain $\alpha(q'_{j+1,i}, q'_{j+1,i-2}) = \alpha(q_{j+1,i}, q_{j+1,i-2})$. Repeating this argument we obtain the claim.

7.6. Proof of Theorem 5.3. From the Yang-Baxter relation for y -variables in Theorem 7.6 we have $R_{M_j}^\epsilon R_{M_{j+1}}^\epsilon R_{M_j}^\epsilon = R_{M_{j+1}}^\epsilon R_{M_j}^\epsilon R_{M_{j+1}}^\epsilon$. By Theorem 7.9, we deduce that

$$R_j^\epsilon R_{j+1}^\epsilon R_j^\epsilon \circ \phi_\epsilon = R_{j+1}^\epsilon R_j^\epsilon R_{j+1}^\epsilon \circ \phi_\epsilon. \quad (7.25)$$

Thus $R_j^\epsilon R_{j+1}^\epsilon R_j^\epsilon = R_{j+1}^\epsilon R_j^\epsilon R_{j+1}^\epsilon$ holds on the image of ϕ_ϵ , and it holds trivially when acting on $\mathbf{q}_1, \dots, \mathbf{q}_{j-1}, \mathbf{q}_{j+3}, \dots, \mathbf{q}_m$. Assuming that $2 \leq j \leq m-3$, it follows easily from the definition of ϕ_ϵ that $R_j^\epsilon R_{j+1}^\epsilon R_j^\epsilon = R_{j+1}^\epsilon R_j^\epsilon R_{j+1}^\epsilon$ holds on \mathcal{Q}_ϵ . Since this identity is ‘local’ in the sense that R_j^ϵ and R_{j+1}^ϵ act only on $\mathbf{q}_j, \mathbf{q}_{j+1}, \mathbf{q}_{j+2}$, we see that the identity holds for all j .

This establishes the braid relations. The involutivity of R_j^ϵ is proved in a similar manner, and the equality $R_i^\epsilon R_j^\epsilon = R_j^\epsilon R_i^\epsilon$ for $|i - j| > 1$ is clear.

8. PROOFS

8.1. Proof of Theorem 3.1. Let A denote the sequence of mutations at $1, 2, \dots, n-2$ in that order. Let A_i denote the first i steps of these mutations. Let B denote the mutations $n-1, n$ followed by the permutation $s_{n-1,n}$. Then the sequence of mutations in $\mathcal{R}_{M,1}$ is exactly $A^{-1} \circ B \circ A$.

Let Q be the initial quiver, restricted to $M^- \cup M \cup M^+$. Let us now describe $Q_i := A_i(Q)$, and set $Q' = Q_{n-2} = A(Q)$. This is described in three levels. We first describe the quiver restricted to M . Then we describe all arrows between M and M^\pm . Lastly, we describe what happens to arrows within $M^+ \cup M^-$.

The following results are proved by a straightforward induction.

Lemma 8.1. *For $i < n-2$, the quiver $Q_i|_M$ consists of:*

- (1) *a possibly empty directed path $1 \rightarrow 2 \rightarrow \dots \rightarrow i$,*
- (2) *an oriented triangle $i \rightarrow n \rightarrow i+1 \rightarrow i$,*
- (3) *an oriented $(n-i)$ -gon $i+1 \rightarrow \dots \rightarrow n \rightarrow i+1$.*

The edge $n \rightarrow i+1$ is mentioned twice, but it only occurs once.

Lemma 8.2. *The quiver $Q'|_M$ consists of*

- (1) *the directed path $1 \rightarrow 2 \rightarrow \dots \rightarrow n-2$,*
- (2) *the edges $n-2 \rightarrow n$ and $n-1 \rightarrow n-2$.*

Lemma 8.3. *Suppose $i \geq 1$. The edges of Q_i between M and M^- are:*

- (1) $1^- \rightarrow 1$,
- (2) $(1 \rightarrow 2^-), (2 \rightarrow 3^-), \dots, ((i-1) \rightarrow i^-), (i \rightarrow (i+1)^-),$

- (3) $(3^- \rightarrow 1), (4^- \rightarrow 2), \dots, ((i+1)^- \rightarrow i-1)$, if $i \geq 2$,
- (4) $((i+2) \rightarrow (i+2)^-), ((i+3) \rightarrow (i+3)^-), \dots, (n \rightarrow n^-)$,
- (5) $((i+3)^- \rightarrow i+2), ((i+4)^- \rightarrow i+3), \dots, (n^- \rightarrow (n-1))$.

Lemma 8.4. *Suppose $i \geq 1$. The edges of Q_i between M and M^+ are:*

- (1) $(n^+ \rightarrow 1)$,
- (2) $(1 \rightarrow 1^+), (2 \rightarrow 2^+), \dots, (i \rightarrow i^+)$,
- (3) $(2^+ \rightarrow 1), (3^+ \rightarrow 2), \dots, (i^+ \rightarrow (i-1))$,
- (4) $((i+2) \rightarrow (i+1)^+), ((i+3) \rightarrow (i+2)^+), \dots, (n^+ \rightarrow (n-1))$,
- (5) $((i+1)^+ \rightarrow (i+1)), ((i+2)^+ \rightarrow (i+2)), \dots, ((n-1)^+ \rightarrow (n-1))$.

Lemma 8.5. *Suppose $i \geq 1$. The edges of Q_i within M^- are:*

- (1) *The directed path $2^- \rightarrow 3^- \rightarrow \dots \rightarrow n^- \rightarrow 1^-$.*

Lemma 8.6. *Suppose $i \geq 1$. The edges of Q_i within M^+ are:*

- (1) *The directed path $1^+ \rightarrow 2^+ \rightarrow \dots \rightarrow n^+$.*

Lemma 8.7. *Suppose $i \geq 1$. The edges of Q_i between M^- and M^+ are:*

- (1) $(1^+ \rightarrow 1^-)$,
- (2) $(2^- \rightarrow n^+)$.

Proposition 8.8. *We have $B(Q') = Q'$. Thus $A^{-1} \circ B \circ A$ sends the quiver Q to itself.*

We now prove that $\mathcal{R}_{M,1}$ acting on the variables x_i is given by the formula R_M from (3.1).

Proposition 8.9. *The effect of A on the cluster variables is given by the formula $x_i \mapsto \tilde{x}_i$, where*

$$\tilde{x}_i = \frac{1}{x_1 x_2 \cdots x_i} \left(x_1^- (x_2 x_3 \cdots x_{i+1}) x_n^+ + \sum_{k=3}^{i+2} x_{k-1}^- (x_k x_{k+1} \cdots x_{i+1} x_n x_1 \cdots x_{k-2}) x_{k-2}^+ \right) \quad (8.1)$$

$$= x_1^- \frac{x_{i+1}}{x_1} x_n^+ + \sum_{k=3}^{i+2} x_{k-1}^- \frac{x_{i+1} x_n}{x_{k-1} x_{k-2}} x_{k-2}^+ \quad (8.2)$$

for $i = 1, 2, \dots, n-2$. Note that there are $(i+1)$ terms in this formula.

It is easy to check that B acting on the variables $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{n-2}, x_{n-1}, x_n$ sends x_{n-1} and x_n to $R_M(x_{n-1})$ and $R_M(x_n)$. Since all mutations act as invertible rational transformations, to prove that $\mathcal{R}_{M,1} = R_M$, it suffices to check that $A \circ R_M = B \circ A$, where A and B are thought of as operators acting on an ordered sequence of n variables. In other words, we need to compute $\tilde{x}_i(R_M(x_1), \dots, R_M(x_n))$.

Let us substitute the formula for R_M into (8.1). Note that $R_M(x_i) = Sx_i$ for some expression S . Since the formula for \tilde{x}_i is homogeneous in x_1, x_2, \dots, x_n , we see that $\tilde{x}_i(R_M(x_1), \dots, R_M(x_n)) = \tilde{x}_i$. This completes the proof of Theorem 3.1.

8.2. Proof of Theorem 3.2. Let

$$S = \frac{\sum_{j=1}^n x_{j+1}^- \left(\prod_{\ell=j+2}^{j-1} x_\ell \right) x_j^+}{\prod_{j=1}^n x_j}. \quad (8.3)$$

Then $R_M(x_i) = Sx_i$. Note that $R_M(S) = S(R_M(x_1), \dots, R_M(x_n)) = S/S^2 = 1/S$. Thus

$$R_M^2(x_i) = S(R_M(x_1), \dots, R_M(x_n))R_M(x_i) = x_i.$$

We have $R_{M^+}(S) = S|_{x_i^+ \mapsto S^+x_i^+} = S^+S$ since S is homogeneous of degree one in the $+$ -variables. Similarly $R_M(S^+) = SS^+$. Thus

$$R_MR_{M^+}R_M(x_i) = R_MR_{M^+}(Sx_i) = R_M(S^+Sx_i) = (SS^+)(1/S)Sx_i = SS^+x_i.$$

Also,

$$R_{M^+}R_MR_{M^+}(x_i) = R_{M^+}R_M(x_i) = R_{M^+}(Sx_i) = SS^+x_i.$$

Similarly, $R_MR_{M^+}R_M(x_i^+) = R_{M^+}R_MR_{M^+}(x_i^+)$. We conclude that $R_MR_{M^+}R_M = R_{M^+}R_MR_{M^+}$.

When M and M' are nonadjacent closed vertical cycles, the equality $R_MR_{M'} = R_{M'}R_M$ is clear.

8.3. Proof of Theorem 3.3. Let \tilde{Q} be the subquiver of $\tilde{Q}_{n,m}$, restricted to $M^- \cup M \cup M^+$ together with all the frozen vertices neighbouring M . Let Q_L be the quiver obtained from \tilde{Q} by removing the vertices corresponding to the frozen variables $X_{i^-,i}$, X_{i,i^+} , $X_{i+1^-,i}$ and X_{i+1,i^+} for $i = 1, \dots, n$. In the same manner, let Q_C be the quiver without the vertices corresponding to the frozen variables $X_{i,i+1}$, $X_{i+1^-,i}$ and X_{i+1,i^+} , and let Q_D be the quiver without the vertices corresponding to the frozen variables $X_{i,i+1}$, $X_{i^-,i}$ and X_{i,i^+} . We write \mathbf{x}_L , \mathbf{x}_C and \mathbf{x}_D for the clusters (including the remaining frozen variables) corresponding to the quivers Q_L , Q_C and Q_D respectively. Similar to the proof in §8.1, we decompose $\tilde{\mathcal{R}}_{M,1} = A^{-1} \circ B \circ A$. The following lemma is easily seen.

Lemma 8.10. *Let $\sigma: (\tilde{Q}, \mathbf{x}) \mapsto \sigma(\tilde{Q}, \mathbf{x})$ be the operation that acts on the mutable vertices by $i^+ \mapsto i+1^-$, $i^- \mapsto i-1^+$, and acts on the frozen vertices by $v_{i^-,i} \mapsto v_{i,i-1^+}$, $v_{i,i-1^+} \mapsto v_{i^-,i}$, $v_{i,i^+} \mapsto v_{i+1^-,i}$, $v_{i+1^-,i} \mapsto v_{i,i^+}$ for all $i = 1, \dots, n$, fixing all other vertices. The action on the cluster variables \mathbf{x} is the natural induced one. Then we have $\sigma(\tilde{Q}, \mathbf{x}) = (\tilde{Q}, \mathbf{x})$, and $\sigma(Q_C, \mathbf{x}_C) = (Q_D, \mathbf{x}_D)$.*

Quiver mutation produces no new arrows between frozen vertices. Thus the contribution of the frozen vertices in a sequence of mutations of (\tilde{Q}, \mathbf{x}) is the superposition of the three distinct contributions in the corresponding mutations of (Q_L, \mathbf{x}_L) , (Q_C, \mathbf{x}_C) and (Q_D, \mathbf{x}_D) .

Proposition 8.11. *We have $\tilde{\mathcal{R}}_{M,1}(Q_*) = Q_*$ for $* = L, C$ and D .*

Proof. We calculate the change of arrows related to the frozen vertices, caused by $\tilde{\mathcal{R}}_{M,1}$, in addition to the change of Q studied in §8.1.

The case of $* = L$: for $i = 1, \dots, n-2$, the arrows related to $X_{i,i+1}$ are changed only by μ_i in A :

$$(i+1 \rightarrow X_{i,i+1} \rightarrow i) \xrightarrow{\mu_i} \begin{cases} (1 \rightarrow X_{1,2}), (X_{1,2} \rightarrow 1^-), (X_{1,2}, n^+) & i = 1, \\ (i \rightarrow X_{i,i+1} \rightarrow i-1) & i = 2, \dots, n-2, \end{cases}$$

and the resulted arrows are invariant under B . For $i = n - 1$, the arrows related to $X_{n-1,n}$ are not affected by A , and invariant under B :

$$\begin{aligned} (n \rightarrow X_{n-1,n} \rightarrow n-1) &\xrightarrow{\mu_{n-1}^{-1}} (n-1 \rightarrow X_{n-1,n}), (n \rightarrow X_{n-1,n}) \\ &\xrightarrow{\mu_n} (n-1 \rightarrow X_{n-1,n} \rightarrow n) \xrightarrow{s_{n-1,n}^{-1,n}} (n \rightarrow X_{n-1,n} \rightarrow n-1). \end{aligned}$$

For $i = n$, the arrows related to $X_{n,1}$ are changed only by μ_1 in A :

$$(1 \rightarrow X_{n,1} \rightarrow n) \xrightarrow{\mu_1} (1^+, X_{n,1}), (2^- \rightarrow X_{n,1}), (X_{n,1} \rightarrow 1),$$

and invariant under B . Then we have $B \circ A(Q_L) = A(Q_L)$.

The case of $* = C$: for $i = 1, \dots, n-2$, the arrows related to $X_{i,i+}$ are changed only by μ_i in A :

$$(i \rightarrow X_{i,i+} \rightarrow i^+) \xrightarrow{\mu_i} (n \rightarrow X_{i,i+}), (i+1^- \rightarrow X_{i,i+}), (X_{i,i+} \rightarrow i). \quad (8.4)$$

It is further changed by B into

$$(i+1^- \rightarrow X_{i,i+}), (n-2 \rightarrow X_{i,i+}), (X_{i,i+} \rightarrow i), (X_{i,i+} \rightarrow n-1), \quad (8.5)$$

for $i = 1, \dots, n-3$, and

$$(n-1^- \rightarrow X_{n-2,n-2+} \rightarrow n-1), \quad (8.6)$$

for $i = n-2$. For $i = n, n-1$, the arrows related to $X_{i,i+}$ are not affected by A but by μ_i and $s_{n-1,n}$ in B as

$$\begin{aligned} (n-1 \rightarrow X_{n-1,n-1+} \rightarrow n-1^+) &\xrightarrow{\mu_{n-1}^{-1}} (X_{n-1,n-1+} \rightarrow n-1), (n^- \rightarrow X_{n-1,n-1+}) \\ &\xrightarrow{s_{n-1,n}^{-1,n} \circ \mu_n} (n^- \rightarrow X_{n-1,n-1+} \rightarrow n). \end{aligned} \quad (8.7)$$

$$\begin{aligned} (n \rightarrow X_{n,n+} \rightarrow n^+) &\xrightarrow{\mu_n} (X_{n,n+} \rightarrow n), (X_{n,n+} \rightarrow n^+), (n-2 \rightarrow X_{n,n+}) \\ &\xrightarrow{s_{n-1,n}^{-1,n}} (X_{n,n+} \rightarrow n-1), (X_{n,n+} \rightarrow n^+), (n-2 \rightarrow X_{n,n+}). \end{aligned} \quad (8.8)$$

The arrows related to $X_{i-,i}$ ($i = 1, \dots, n-2$) are changed by $\mu_i, \mu_{i+1}, \dots, \mu_{n-2}$ in A . For $i = 1$ and $j = 1, \dots, n-2$ we have

$$(1^- \rightarrow X_{1-,1} \rightarrow 1) \xrightarrow{\mu_j \circ \dots \circ \mu_1} (j \rightarrow X_{1-,1}), (X_{1-,1} \rightarrow j+1), (X_{1-,1} \rightarrow n^+), \quad (8.9)$$

and for $i = 2, \dots, n-2$ and $j = i, \dots, n-2$ we have

$$(i^- \rightarrow X_{i-,i} \rightarrow i) \xrightarrow{\mu_j \circ \dots \circ \mu_i} (i^- \rightarrow X_{i-,i}), (j \rightarrow X_{i-,i}), (X_{i-,i} \rightarrow i-1), (X_{i-,i} \rightarrow j+1). \quad (8.10)$$

Furthermore, the action of B changes (8.9) into

$$(n \rightarrow X_{1-,1} \rightarrow n^+) \quad (8.11)$$

and (8.10) of $j = n-2$ is changed into

$$(i^- \rightarrow X_{i-,i}), (n \rightarrow X_{i-,i}), (X_{i-,i} \rightarrow i-1). \quad (8.12)$$

The arrows related to $X_{i-,i}$ ($i = n-1, n$) are not affected by A but by μ_i and $s_{n-1,n}$ in B :

$$\begin{aligned} (n-1^- \rightarrow X_{n-1^-,n-1} \rightarrow n-1) \\ \xrightarrow{\mu_{n-1}^{-1}} (n-1^- \rightarrow X_{n-1^-,n-1}), (n-1, X_{n-1^-,n-1}), (X_{n-1^-,n-1}, n-2) \\ \xrightarrow{s_{n-1,n} \circ \mu_n} (n-1^- \rightarrow X_{n-1^-,n-1}), (n, X_{n-1^-,n-1}), (X_{n-1^-,n-1}, n-2), \end{aligned} \quad (8.13)$$

$$(n^- \rightarrow X_{n^-,n} \rightarrow n) \xrightarrow{\mu_n} (n \rightarrow X_{n^-,n} \rightarrow n-1^+) \xrightarrow{s_{n,n}^{-1}} (n-1 \rightarrow X_{n^-,n} \rightarrow n-1^+). \quad (8.14)$$

We can check $B \circ A(Q_C) = A(Q_C)$ by comparing (8.4)–(8.8) and (8.9)–(8.14) in the following way.

The arrows (8.12) with $i = 2$ in $B \circ A(Q_C)$ corresponds to the last part of (8.4) with $i = 1$ in $A(Q_C)$. Similarly, (8.12) ($i = 3, \dots, n-2$) and the last part of (8.13) in $B \circ A(Q_C)$ respectively corresponds to the last part of (8.4) ($i = 2, \dots, n-2$) in $A(Q_C)$. Eq. (8.11) corresponds to the first part of (8.8), and (8.14) corresponds to the first part of (8.7). Here we see that the frozen variable $X_{i-,i}$ appearing in $B \circ A(Q_C)$ corresponds to $X_{i-1,i-1^+}$ appearing in $A(Q_C)$, from which it follows that $\tilde{\mathcal{R}}_{M,1}$ replaces $X_{i-1,i-1^+}$ with $X_{i-,i}$ in \mathbf{x}_L .

Moreover, (8.5) ($i = 1, \dots, n-3$) in $B \circ A(Q_C)$ respectively corresponds to the last part of (8.10) ($i = 2, \dots, n-2$) in $A(Q_C)$ with $j = n-2$. Eq. (8.6) and (8.7) in $B \circ A(Q_C)$ respectively corresponds to the first part of (8.13) and (8.14) in $B(Q_C)$. The last part of (8.8) in $B \circ A(Q_C)$ corresponds to that of (8.9) with $j = n-2$ in $A(Q_C)$. Here we see that the frozen variable $X_{i,i+}$ appearing in $B \circ A(Q_C)$ corresponds to $X_{i+1^-,i+1}$ appearing in $A(Q_C)$, from which it follows that $\tilde{\mathcal{R}}_{M,1}$ replaces $X_{i+1^-,i+1}$ with $X_{i,i+}$.

The case of $* = D$ follows from the case of $* = C$ and Lemma 8.10. □

From the change of arrows studied in the above proof, the following lemmas are obtained.

Lemma 8.12. *For the cluster variables x_i in \mathbf{x}_L , we have that $\tilde{x}_i = A(x_i)$ is given by*

$$\tilde{x}_i = \sum_{k=0}^i x_{k+1}^- \frac{x_{i+1} x_n}{x_{k+1} x_k} x_k^+ X_{k,k+1}; \quad i = 1, 2, \dots, n-2.$$

Furthermore, B changes x_{n-1} and x_n into

$$\begin{aligned} \tilde{x}_{n-1} &= \frac{\tilde{x}_{n-2}}{x_n} + \frac{x_n^- x_{n-1}^+}{x_n} X_{n-1,n}, \\ \tilde{x}_n &= \frac{\tilde{x}_{n-2}}{x_{n-1}} + \frac{x_n^- x_{n-1}^+}{x_{n-1}} X_{n-1,n}. \end{aligned}$$

Lemma 8.13. *For the cluster variables x_i in \mathbf{x}_C , we have that $\tilde{x}_i = A(x_i)$ is given by*

$$\tilde{x}_i = \sum_{k=0}^i x_{k+1}^- \frac{x_{i+1} x_n}{x_{k+1} x_k} x_k^+ \prod_{\ell=1}^k X_{\ell^-, \ell} \prod_{\ell=k+1}^i X_{\ell, \ell^+}; \quad i = 1, 2, \dots, n-2. \quad (8.15)$$

Furthermore, B changes x_{n-1} and x_n into

$$\tilde{x}_{n-1} = \frac{\tilde{x}_{n-2}}{x_n} X_{n-,n} + \frac{x_n^- x_{n-1}^+}{x_n} \prod_{\ell \neq n-1} X_{\ell, \ell^+}, \quad (8.16)$$

$$\tilde{x}_n = \frac{\tilde{x}_{n-2}}{x_{n-1}} X_{n-1-,n-1} + \frac{x_n^- x_{n-1}^+}{x_{n-1}} \prod_{\ell \neq n} X_{\ell^-, \ell}. \quad (8.17)$$

Proof. From (8.4) and (8.9), we recursively obtain

$$\begin{aligned} \tilde{x}_1 &= \frac{x_1^- x_2 x_n^+ X_{1,1^+}}{x_1} + \frac{x_2^- x_n x_1^+ X_{1^-,1}}{x_1}, \\ \tilde{x}_i &= \frac{\tilde{x}_{i-1} x_{i+1} X_{i,i^+}}{x_i} + \frac{x_{i+1}^- x_n x_i^+ \prod_{\ell=1}^i X_{\ell^-, \ell}}{x_i}, \end{aligned} \quad (8.18)$$

for $i = 2, \dots, n-2$, then (8.15) follows. The expression of \tilde{x}_{n-1} and \tilde{x}_n are obtained from (8.4)–(8.14). \square

Proposition 8.14. *We have $\tilde{\mathcal{R}}_{M,1}(Q_*, \mathbf{x}_*) = (Q_*, \tilde{R}_{M,*}(\mathbf{x}))$ for $*$ = L, C and D , where $\tilde{R}_{M,*}(\mathbf{x})$ is obtained from $\tilde{R}_M(\mathbf{x})$ (3.4), (3.5) by setting all frozen variables in $\mathbf{x} \setminus \mathbf{x}_*$ to be 1.*

Proof. As in the proof of Theorem 3.1, it is enough to prove that $A \circ \tilde{R}_{M,*} = B \circ A$. In this proof, for simplicity we write X_{i+} and X_{i-} for X_{i,i^+} and $X_{i-,i}$ respectively.

Case $*$ = L : From Lemma 8.12, it is easy to check that $\tilde{x}_{n-1} = \tilde{R}_{M,L}(x_n)$ and $\tilde{x}_n = \tilde{R}_{M,L}(x_{n-1})$. For $i = 1, \dots, n$ we have $\tilde{R}_{M,L}(x_i) = S' x_i$ where S' is a Laurent polynomial of the cluster variables in \mathbf{x}_L independent of i . Then we obtain $A \circ \tilde{R}_{M,L} = B \circ A$, in the same manner as Theorem 3.1.

Case $*$ = C : For $i = n-1, n$, it is easy to check that $\tilde{x}_i = \tilde{R}_{M,C}(x_i)$ by using (8.16) and (8.17). Define $\tilde{A}_i := \mu_i \circ \mu_{i+1} \circ \dots \circ \mu_{n-2}$, the first $n-1-i$ steps of the $n-2$ mutations in A^{-1} . Let x'_i be the cluster variable given by $\tilde{x}_i \mapsto x'_i$ by the action of \tilde{A}_i . In the following we show that $\tilde{R}_{M,C}(x_i) = x'_i$ by induction on i , from $i = n-2$ to 1. By (8.18), for $i = 1, \dots, n-2$ we have

$$x'_i = \frac{\tilde{x}_{i-1} x'_{i+1} X_{i+1-}}{\tilde{x}_i} + \frac{x_{i+1}^- \tilde{x}_n x_i^+ X_{n+} X_{1+} \dots X_{i-1+}}{x'_i}, \quad (8.19)$$

where we denote $x'_{n-1} = \tilde{x}_{n-1}$. Here we have used that $\tilde{A}_i \circ B \circ A(Q_C) = \mu_{i-1} \circ \dots \circ \mu_1(Q_C)$ and that the frozen variables X_{i+} and X_{i+1-} are interchanged by $B \circ A$. When $i = n-2$, (8.19) includes $\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_{n-3}$. From (8.18) of $i = n-2$, (8.16) and (8.17) respectively, we obtain the expression of $\tilde{x}_{n-3}, \tilde{x}_{n-1}$ and \tilde{x}_n in terms of \tilde{x}_{n-2} . We substitute them into (8.19), and show $x'_{n-2} = \tilde{R}_{M,C}(x_{n-2})$. Now, assume that $x'_j = \tilde{R}_{M,C}(x_j)$ for $j = i+1, \dots, n-3$. Then we have

$$x'_i \tilde{x}_i = \tilde{x}_{i-1} \tilde{R}_{M,C}(x_{i+1}) X_{i+1-} + x_{i+1}^- \tilde{x}_n x_i^+ X_{n+} X_{1+} \dots X_{i-1+}.$$

By using (8.18) we erase \tilde{x}_{i-1} in the above, and what we have to show is

$$\begin{aligned} & \left(\tilde{R}_{M,C}(x_i) x_{i+1} X_{i+} - \tilde{R}_{M,C}(x_{i+1}) x_i X_{i+1-} \right) \tilde{x}_i \\ &= x_{i+1}^- x_i^+ \left(\tilde{R}_{M,C}(x_n) x_{i+1} X_{n+} X_{1+} \dots X_{i+} - \tilde{R}_{M,C}(x_{i+1}) x_n X_{1-} \dots X_{i+1-} \right). \end{aligned}$$

Both sides turn out to be

$$x_{i+1}^- x_i^+ \tilde{x}_i \left(\prod_{\ell=1}^n X_{\ell^+} - \prod_{\ell=1}^n X_{\ell^-} \right),$$

and we obtain $\tilde{R}_{M,C}(x_i) = x'_i$.

The case of $* = D$ follows from the case of $* = C$ and Lemma 8.10. \square

To complete the proof of Theorem 3.3, we use the following fundamental result of Lee and Schiffler [LS], which we formulate for skew-symmetric cluster algebras of geometric type.

Theorem 8.15. *Let $(x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_{n+m})$ be the cluster variables of a seed in a skew-symmetric cluster algebras of geometric type, where x_1, \dots, x_n are the mutable variables and x_{n+1}, \dots, x_{n+m} are the frozen variables. Let y be any other cluster variable. Then y is a Laurent polynomial in $\mathbb{Z}[x_1^{\pm 1}, \dots, x_{n+m}^{\pm 1}]$ with nonnegative integer coefficients.*

By Theorem 3.1, we know that

$$\tilde{\mathcal{R}}_{M,1}(x_i)|_{\mathbf{X} \rightarrow 1} = \frac{\sum_{j=1}^n x_{j+1}^- \left(\prod_{\ell=j+2}^{j-1} x_{\ell} \right) x_j^+}{\prod_{j \neq i} x_j}$$

when all the frozen variables are set to 1. All the n (Laurent) monomials in this formula are distinct and have coefficient one. It follows from Theorem 8.15 that the formula for $\tilde{\mathcal{R}}_{M,1}(x_i)$ with frozen variables is of the form

$$\tilde{\mathcal{R}}_{M,1}(x_i) = \frac{\sum_{j=1}^n x_{j+1}^- \left(\prod_{\ell=j+2}^{j-1} x_{\ell} \right) x_j^+ \beta_j}{\prod_{j \neq i} x_j}$$

for some Laurent monomials β_j in the frozen variables \mathbf{X} . Theorem 3.3 then follows from Proposition 8.14.

8.4. Proof of Theorem 3.4. The equality $\tilde{R}_M^2 = \text{id}$ follows from the fact that cluster mutations are involutive. The commutativity $\tilde{R}_{M_i} \tilde{R}_{M_j} = \tilde{R}_{M_j} \tilde{R}_{M_i}$ for $|i - j| > 1$ is clear from the definitions.

The equality $\tilde{R}_{M_j} \tilde{R}_{M_{j+1}} \tilde{R}_{M_j} = \tilde{R}_{M_{j+1}} \tilde{R}_{M_j} \tilde{R}_{M_{j+1}}$ follows from the braid relations of Theorem 7.6, evaluated at $\epsilon = 1$. To see this, we note that the braid relation of Theorem 7.6 can be interpreted as saying that some particular I -sequence \mathbf{i} is a σ -period for a y -seed $(Q_{n,m}, \mathbf{y})$, where σ is a permutation of the vertices of $Q_{n,m}$, in the sense of (any of the equivalent statements of) Theorem 2.1. Here, \mathbf{i} is the sequence of mutations corresponding to $\tilde{R}_{M_j} \tilde{R}_{M_{j+1}} \tilde{R}_{M_j} \tilde{R}_{M_{j+1}} \tilde{R}_{M_j} \tilde{R}_{M_{j+1}}$.

It follows from Theorem 2.2 that \mathbf{i} is also a σ -period of $(B, \mathbf{x}, \mathbf{y})$, where $Q_{n,m} = Q(B)$ and \mathbf{y} lives in the tropical semifield $\mathbb{P}_{\text{trop}}(\mathbf{y})$. By definition, the mutation of x -variables in $(B, \mathbf{x}, \mathbf{y})$ -seeds is exactly the mutation for the principal coefficient cluster algebra with exchange matrix B [FZ07]. It follows from [FZ07, Theorem 3.7] that \mathbf{i} is a σ -period for the initial seed of any skew-symmetric cluster algebra of geometric type with exchange matrix B , that is, for any cluster algebra obtained from adding frozen variables to (B, \mathbf{x}) . In particular, this applies to the cluster algebra associated to $\tilde{Q}_{n,m}$, where we consider all variables \mathbf{X} frozen.

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